Online Appendix to:

“Learning from Prices: Public Communication and Welfare”

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Contents

I Co-movements between Aggregate Shocks and Output

The assumption that money is separable from consumption and leisure in the utility function of the household is a useful benchmark, because it implies that, under perfect information, any type of nominal disturbance (that is, any shocks to M or V) would have no real effects. We now show that under imperfect information, as in Lucas (1972), workers are confused regarding the underlying shocks, and nominal disturbances now have real effects.

Using the fact that $y = v - p$ and the result of Proposition 1, we have:

$$y = \frac{\delta}{1+\delta}\frac{\Psi_v}{\Psi}v + \left(1 - \frac{\delta}{1+\delta}\frac{\Psi_v}{\Psi}\right)\frac{\theta}{\Omega} - K_0,$$

and given that $\delta\Psi_v < (1+\delta)\Psi$, velocity and productivity shocks are expansionary.

Proposition I.1. *In a linear equilibrium, the covariance between v and y and the covariance between θ and y are both strictly positive. Both of these covariances are also higher than under full information.*

Under full information, velocity shocks do not affect aggregate output; thus, their covariance with output is zero. Proposition I.1 shows that, under imperfect information, velocity shocks have real effects. When a positive velocity shock is realized, final and intermediate goods prices increase. However, because of the aggregate technology shock, θ, workers cannot infer the exact realization of the velocity shock from these price increases. This generates a version of the confusion problem in Lucas (1972): workers do not attribute the full extent of the increase in prices to the velocity shock, so they mistakenly infer that part of the observed increase in the intermediate price signal, $v + a_i$, is caused by an increase in local demand, a_i. This boosts their labor supply and increases aggregate output.

Under full information, aggregate output should respond one-to-one to shocks on the technology parameter θ (as, given our specification, these aggregate productivity shocks do not affect labor supply). Proposition I.1 reveals that, under imperfect information, output is more responsive to technological shocks. Indeed, when a positive productivity shock occurs, workers attribute some of the resulting decrease in prices to a fall in velocity. But given that the signal received by the workers, $v + a_i$, has not changed, they infer that local demand is relatively high, which generates an expansion of labor supply. The effect on aggregate output thus will be bigger under partial information because aggregate labor is now responding positively to a technology shock.

Proof of Proposition I.1

In the general CRRA-CES specification, we use that $\gamma y = v - p$ and get:

$$\begin{aligned} y =& \frac{\delta}{(1+\gamma\delta)} \frac{\bar{\Psi}_v}{\Psi} v + \left(\frac{1+\delta}{1+\gamma\delta} + \frac{\delta}{1+\gamma\delta} \frac{\Omega\bar{\Psi}_\theta}{\Psi} \right) \theta - \delta \frac{\Psi_\theta - \bar{\Psi}_\theta}{(1+\gamma\delta)\Psi} \Omega \eta_\theta - \delta \frac{\Psi_v - \bar{\Psi}_v}{(1+\gamma\delta)\Psi} \eta_v \\ & + \frac{\delta + \phi}{2(\phi - 1)(1+\gamma\delta)} \Xi + \frac{\delta}{2(1+\gamma\delta)} \left(\frac{1}{\Psi} - \frac{1}{\psi_a} \right) \end{aligned} \tag{I.1}$$

which implies that the correlation between y and v, or between y and θ, are strictly positive.

Note that the correlation between velocity shocks and output under perfect information is zero (as can be seen above by taking the limit as $\Psi \to \infty$). Similarly, the correlation between productivity shocks and output is bigger than under full information, as under full information the elasticity of output with respect to the productivity shock is $(1+\delta)/(1+\gamma\delta)$.

Letting $\gamma = 1$, $\phi = 1$ and $\bar{\Psi}_v = \Psi_v$ and $\bar{\Psi}_\theta = \Psi_\theta$, delivers the result.

II Macro versus Micro Labor-supply Elasticity

Proposition 4 shows that public information is welfare reducing only if the labor supply, δ, is sufficiently high. There is lot of disagreement about the value of this parameter: empirical micro estimates result in low values of the elasticity, while macro studies generate much larger values.

II.1 An Extension with Frictional Unemployment

We will show below that it is this larger macro elasticity that matters for information aggregation and for the welfare effects of public announcements. To make this point we introduce here a simple extension of our model with frictional unemployment, where the micro and macro elasticity are allowed to differ: the micro elasticity controls the intensive margin (the choice of hours worked per job) while the macro elasticity also incorporates the extensive margin (the number of filled jobs). We show that the elasticity that matters for learning and welfare is the macro elasticity: that is, how *total hours* per sector respond to shocks will determine the sensitivity of the equilibrium prices to private information, and ultimately the way that workers learn from each other. In particular, if one were to use the micro elasticity instead of the macro elasticity in our key equilibrium equation, then one would obtain the wrong equilibrium equations and the wrong welfare prescriptions.

We make the following change to our basic model: we assume that the representative family is extended to encompass a continuum of intermediate good producers in each sector. In order to produce intermediate goods, workers in each sector need to be matched with intermediate goods producers, and these producers can post vacancies at a utility cost $\kappa > 0$ in their sectors. Let V_i denote the measure of vacancies posted in sector i, and let N_i denote the number of employed workers in sector i. Then the utility of the representative family is:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \Big(\log C_t + \frac{1}{V} \log \frac{M_{t-1}^d}{P_t} - \frac{\delta}{1+\delta} \int_0^1 N_{it} L_{it}^{1+\frac{1}{\delta}} \, di - \kappa \int_0^1 V_i \, di \Big) \right],$$

subject to the same budget constraint as before, except that the real value of intermediate good production is now equal to $\int P_{it} \Theta_i N_{it} L_{it} / P_t \, di$. Again, we will concentrate in the first period and drop the time subscript. First, note that the equilibrium equations (1), (2), and (3) remain unaltered, as do their log linearized versions (7), (8), and (9). Thus, all that is left to be determined is the new supply equation for intermediate good.

We assume that the matching process occurs within a given intermediate good sector. To simplify the analysis we further assume that matches last only one period.[1] Let the probability that a vacancy matches with a worker in sector i be $q(V_i) \equiv \mu V_i^{-\alpha}$, where $\alpha \in (0, 1)$ and $\mu > 0$.[2] A law-of-large-numbers argument will imply that the total number of matches between workers and vacancies in sector i is:

$$N_i = V_i q(V_i) = \mu V_i^{1-\alpha}. \tag{II.2}$$

When a worker and an intermediate good producer meet, they negotiate over the number of hours worked, L_i, and over compensation. We assume that both workers and intermediate good producers share the same information set, and seek to maximize their contribution to household welfare. Because of identical information sets and marginal utilities for money balances, $\mathbb{E}_i[\lambda/P]$, the surplus of one-period long worker-producer relationship is:

$$\mathbb{E}_i \left[\frac{\lambda}{P} P_i \Theta_i L_i \right] - \frac{\delta}{1+\delta} L_i^{1+\frac{1}{\delta}}.$$

[1]This assumption preserves the static nature of our model and leads to closed-form solutions for the number of filled jobs. Clearly, in a quantitative analysis, it would be important to relax it.

[2]If the measure of vacancies is low, this matching function has the unpleasant implication that $q(V_i) \geq 1$, i.e. the matching probability is greater than 1. Keep in mind, however, that we can always pick parameter values so that, in equilibrium, $q(V_i) \leq 1$ in most sectors.

Based on Hall's (2009) argument that firms and workers should choose the amount of hours efficiently, it follows that L_i is chosen to maximize the surplus:

$$L_i^{\frac{1}{\delta}} = \mathbb{E}_i \left[\frac{\lambda}{P} \right] P_i \Theta_i = \mathbb{E}_i \left[\frac{\beta}{1-\beta} \frac{1}{MV} \right] P_i \Theta_i, \tag{II.3}$$

where, as before, the second equality follows from the quantity equation (2).

Assuming a generalized Nash-Bargaining over compensation, the payment received by the firm is some share $\rho \in [0, 1]$ of the surplus:

$$\rho \left(\mathbb{E}_i \left[\frac{\lambda}{P} \right] P_i \Theta_i L_i - \frac{\delta}{1+\delta} L_i^{1+\frac{1}{\delta}} \right) = \frac{\rho}{1+\delta} L_i^{1+\frac{1}{\delta}},$$

where the equality follows after substituting in the surplus maximizing labor supply as shown in equation (II.3).

Intermediate goods producers will post infinite vacancies (or zero) if their expected compensation per vacancy is above (or below) the vacancy creation cost κ. Thus, in equilibrium:

$$\kappa = \mu V_i^{-\alpha} \frac{\rho}{1+\delta} L_i^{1+\frac{1}{\delta}}. \tag{II.4}$$

From equations (II.2) and (II.4), and after normalizing $\mu^{\frac{1}{1-\alpha}} \rho = \kappa(1+\delta)$, we find that $N_i = L_i^{\frac{1+\delta}{\alpha\delta}(1-\alpha)}$.[3] Total production of intermediate i then is $Y_i = \Theta_i N_i L_i$, which after using (II.3), yields a supply equation for the intermediate good:

$$Y_i = \Theta_i \Big(\mathbb{E}_i \Big[\frac{\beta}{1-\beta} \frac{1}{MV} \Big] \Theta_i P_i \Big)^{\delta + \frac{(1-\alpha)(1+\delta)}{\alpha}},$$

that is identical to equation (4) except for a different Frisch elasticity parameter, $\hat{\delta}$:

$$\hat{\delta} \equiv \delta + \frac{(1-\alpha)(1+\delta)}{\alpha} > \delta.$$

Given that the remaining equilibrium equations, (1), (2), and (3), do not change, the characterization of linear equilibria is the same as in Proposition 1, but replacing the micro labor supply elasticity parameter, δ, with the higher macro elasticity parameter defined above, $\hat{\delta}$. Importantly, we show below that the same is also true for welfare:

Proposition II.1. *In the economy with frictional unemployment, public information increases*

[3] This normalization is without loss of generality for the results that follow.

ex-ante utilitarian welfare if and only if it increases the posterior precision about v, $\psi_a + \Omega_\star^2 \psi_\theta + \Psi_v + \Omega_\star^2 \Psi_\theta$, where $\Omega_\star$ is the highest fixed point of:

$$\Omega = \frac{1}{1+\hat{\delta}} + \frac{\hat{\delta}}{1+\hat{\delta}} \frac{\psi_a + \Omega^2 \psi_\theta}{\Psi_v + \Omega^2 \Psi_\theta + \psi_a + \Omega^2 \psi_\theta} \qquad \textit{where} \quad \hat{\delta} \equiv \delta + \frac{(1-\alpha)(1+\delta)}{\alpha} > \delta.$$

Proof of Proposition II.1

The only thing left to show is that the welfare analysis is the same as before. Clearly, the expected utility of output and real balance is the same as before, again after replacing δ by $\hat{\delta}$. Let us turn, then, to the remaining term in the utilitarian welfare function:

$$-\mathbb{E}\left[\frac{\delta}{1+\delta}\int_0^1 N_i L_i^{1+\frac{1}{\delta}}\, di + \kappa \int_0^1 V_i\, di\right]. \tag{II.5}$$

First, note that, in equilibrium,

$$N_i = L_i^{\frac{(1-\alpha)(1+\delta)}{\alpha\delta}} \Rightarrow N_i L_i^{1+\frac{1}{\delta}} = L_i^{\frac{1+\delta}{\alpha\delta}}.$$

At the same time,

$$N_i = \mu V_i^{1-\alpha} \Rightarrow \kappa V_i = \mu^{-\frac{1}{1-\alpha}} \kappa L_i^{\frac{1+\delta}{\alpha\delta}} = \frac{\rho}{1+\delta} L_i^{\frac{1+\delta}{\alpha\delta}}.$$

Thus, (II.5) is equal to:

$$-\frac{\rho+\delta}{1+\delta}\mathbb{E}\left[\int_0^1 N_i L_i^{1+\frac{1}{\delta}}\right]$$

Now, proceeding as before, we can write that:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1 N_i L_i^{1+\frac{1}{\delta}}\, di\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[N_i L_i^{1+\frac{1}{\delta}} \mid v, \theta\right]\right] = \mathbb{E}\left[N_i L_i^{1+\frac{1}{\delta}}\right] = \mathbb{E}\left[N_i L_i \mathbb{E}_i\left[\lambda \frac{P_i}{P}\Theta_i\right]\right] \\
=&\mathbb{E}\left[N_i L_i \lambda \frac{P_i}{P}\Theta_i\right] = \mathbb{E}\left[Y_i Y^{-\gamma}\frac{\partial Y}{\partial Y_i}\right] = \mathbb{E}\left[Y^{-\gamma}\mathbb{E}\left[Y_i \frac{\partial Y}{\partial Y_i} \mid v, \theta\right]\right] \\
=&\mathbb{E}\left[Y^{-\gamma}\int_0^1 Y_i \frac{\partial Y}{\partial Y_i}\, di\right] = \mathbb{E}\left[Y^{1-\gamma}\right].
\end{aligned}$$

So, (II.5) is equal to:

$$-\frac{\rho+\delta}{1+\delta}\mathbb{E}\left[Y^{1-\gamma}\right],$$

and the rest of the analysis goes through.

III Bond market

A familiar way in which an economy aggregates dispersed private information is through financial markets. One might wonder, then, how robust the results regarding the social value of public announcements that we have obtained are to the introduction of a financial market where households from different locations can interact. To answer this question, we introduce what we believe is a natural financial market in our economy: households are allowed to trade a nominal bond in zero net supply when deciding consumption, and the price of such bond is observed by all agents. Our main result is that the equilibrium nominal interest rate in the financial market does not provide any new information to the workers and that the allocation obtained by a competitive equilibrium when the bond market is closed remains the allocation of a competitive equilibrium once it is opened.

Thus, suppose that the shopper at period t can buy a bond that pays a unit of the currency in the following period, $t+1$, and let us denote by Q_t its nominal price. The budget constraint of the household i in period t is now given by

$$C_t + \frac{M_t^d}{P_t} + \frac{B_t}{P_t}Q_t \leq \int_0^1 \frac{P_{it}}{P_t}\Theta_i L_{it} + \frac{M_{it-1}^d}{P_{it}} + \frac{B_{t-1}}{P_{it}}, \tag{III.6}$$

where B_t is the amount of the bond held by the representative family. The bond market clearing condition imposes that $B_t = 0$ for all $t \geq 0$.

We now check that the allocation without a nominal bond market remains an equilibrium once the nominal bond market opens. Because the shopper is acting under full information, her Euler equation is:

$$Q_t = \beta\frac{u'(C_{t+1})}{u'(C_t)}\frac{P_t}{P_{t+1}} = \beta\frac{\lambda_{t+1}}{P_{t+1}}\frac{P_t}{\lambda_t},$$

where, from the first-order conditions, $u'(C_t) = \lambda_t$. From the equation (2) it then follows that

$$Q_t = \beta$$

Note that the price of the bond does not reveal any information: it is just equal to the discount factor. Thus, any equilibrium allocation when the bond market is closed remains an equilibrium when the bond market is open with $B_t = 0$ and $Q_t = \beta$.

IV Large Idiosyncratic Shocks

A recent literature has documented that idiosyncratic variations in prices (and other quantities) are an order-of-magnitude bigger than aggregate variations (see, among others, Bils and Klenow, 2004). Hence, one may argue that the main concern of economic agents is to optimally respond to their idiosyncratic shocks and, as a result, that forecasting macroeconomic shocks may not matter much for their welfare. In this section we show that contrary to this view, our welfare results are robust to arbitrarily large levels of idiosyncratic price volatility.

Suppose that the productivity parameter now includes a perfectly observed and purely idiosyncratic component. That is, the productivity in sector i is now $\hat{\Theta}_i = \Theta_i \exp(g_i)$ where Θ_i is as before and where g_i is an idiosyncratic shock generated from a normal distribution with mean zero and constant variance and independent across sectors. We assume that the individual realization of g_i is observed perfectly by workers in location i. The equilibrium equations (7), (8) and (9) are unchanged, and only the supply of intermediate goods is affected: in equation (10), $\hat{\theta}_i$ replaces θ_i. An similar argument to the one in the paper implies that prices are:

$$p_i = \frac{1}{1+\delta}\left(v + a_i - \frac{1}{2\psi_a}\right) + \frac{\delta}{1+\delta}\left(\mathbb{E}_i\left[v\right] - \frac{V_i\left[v\right]}{2}\right) - \theta_i - g_i. \qquad \text{(IV.7)}$$

Note that a higher variance of g_i will generate higher cross-sectional dispersion of prices (given expectations). However, given that g_i is known to worker i, and that it has a cross-sectional average equal to zero, its presence in the intermediate good price equation has no effect on the signal extraction problem that workers face, and thus would not affect the resulting equilibrium fixed-point equation nor the cross-sectional dispersion of expectations about the velocity shock. It then follows that any level of idiosyncratic variation can be added to the model by changing the variance of g_i without affecting our previous analysis, including the welfare result stated in Section 5. We interpret this as follows: the fact that there is a large idiosyncratic component to the individual decisions does not imply that the welfare effects of announcements regarding the underlying aggregate shocks are necessarily trivialized. In particular, in the present example, idiosyncratic volatility does not affect the welfare calculations at all.

V A Full Analysis of the Fixed-point Equation

For a given precision vector $(\Psi_\theta, \Psi_v, \psi_\theta, \psi_v)$, we let $\mathcal{M}$ be, for some fixed vector of the private information precisions (ψ_θ, ψ_v), the set of (Ψ_θ, Ψ_v) such that there are multiple (more than one) equilibria. We start by noting that the set function $\mathcal{M}$ is homogeneous of degree one: when (ψ_θ, ψ_v) is scaled by some $\mu > 0$, the set $\mathcal{M}$ is scaled by the same constant. Indeed scaling *all* precision parameters $(\psi_\theta, \psi_v, \Psi_\theta, \Psi_v)$ up and down by the same constant does not change the fixed point equation for $\Omega_\star$, so it does not change the set of equilibrium $\Omega_\star$. This means that multiplicity does not depend on the absolute level of information, but on how the information is divided among the different sources.

Our main proposition is:

Proposition V.1. *For all (ψ_v, ψ_θ) such that*

$$\left(\frac{\delta}{1+\delta}\right)^2 \frac{\psi_\theta}{\psi_v}\left[\left(1 - 2\frac{1}{\delta\gamma}\right)^3 - 27\left(\frac{1}{\delta\gamma}\right)^2\right] > 27, \tag{V.8}$$

the set $\mathcal{M}$ is delimitated by continuous, strictly positive, strictly decreasing upper and lower boundaries

$$\begin{aligned} U(\Psi_\theta) &= \max\{\Psi_v : (\Psi_\theta \Psi_v) \in \mathcal{M}\} \\ L(\Psi_\theta) &= \min\{\Psi_v : (\Psi_\theta, \Psi_v) \in \mathcal{M}\}, \end{aligned}$$

defined for all Ψ_θ less than some upper bound Ψ_θ^{max} satisfying $L(\Psi_\theta^{max}) = U(\Psi_\theta^{max})$.

This proposition is illustrated in Figure 1, which shows the set $\mathcal{M}$ in the (Ψ_θ, Ψ_v) plane. Multiple equilibria arise when (Ψ_θ, Ψ_v) lies in between the two boundaries. The boundaries are strictly positive. Thus, when (Ψ_θ, Ψ_v) is small enough, there is a unique equilibrium and a mild increase in the public information vector may create multiplicity.

Figure 2 shows graphically how changing public information distorts the fixed-point equation and can create multiple equilibria. For a low value of Ψ_θ, complementarities are strong and the right-hand-side of the fixed-point equation has a sharply increasing

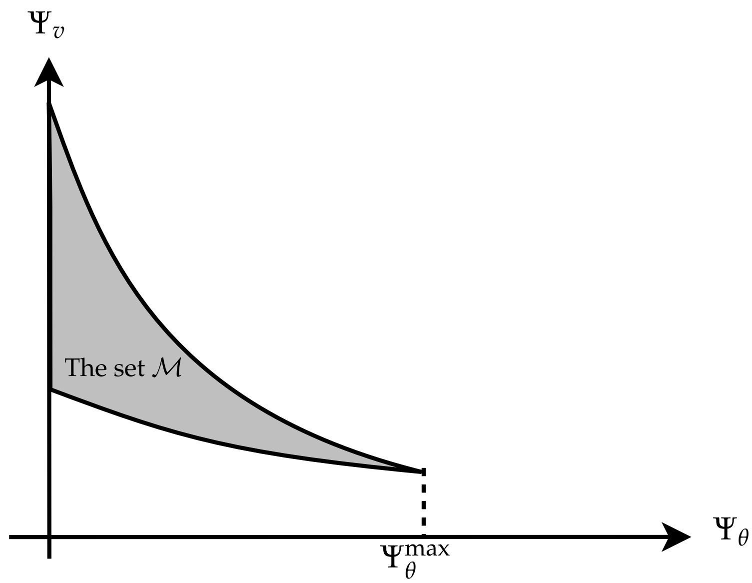

Figure 1: The set $\mathcal{M}$ and its boundaries.

S-shape; it has a unique intersection with the 45-degree line in the upper branch of the S. For a middle value of Ψ_θ, the S-shape rises more slowly and three intersections arise. When Ψ_θ is large, the S becomes decreasing and a unique intersection is obtained. A similar graphical analysis applies to changes in Ψ_v.

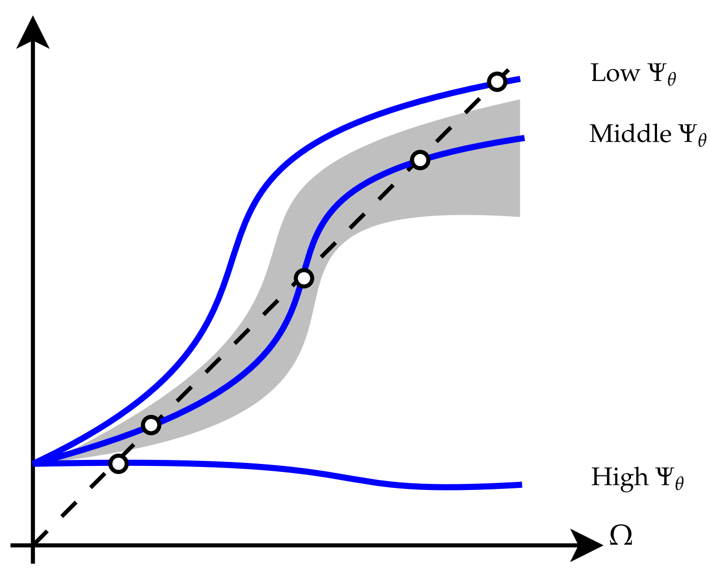

Figure 2: The impact of changing Ψ_θ on the fixed-point equilibrium equation.

The following proposition accounts for all equilibria:

Proposition V.2. *Given* (ψ_θ, ψ_v) *satisfying (V.8), then*

1. *for all* (Ψ_θ, Ψ_v) *in the interior of* $\mathcal{M}$, *there are three equilibria,* $\Omega_L < \Omega_M < \Omega_H$*;*

2. *for all* (Ψ_θ, Ψ_v) *either on the upper or lower boundaries of* $\mathcal{M}$, *there are two distinct equilibria,* $\Omega_L < \Omega_H$*; for* $(\Psi_\theta^{max}, U(\Psi_\theta^{max}))$*, there is only one;*

3. *as* (Ψ_θ, Ψ_v) *approaches the lower boundary from below, the unique equilibrium converges to* Ω_H*;*

4. *as* (Ψ_θ, Ψ_v) *approaches the upper boundary from above, the unique equilibrium converges to* Ω_L*.*

Thus, in the interior of the multiple equilibrium region, $\mathcal{M}$, there are three equilibria. Note that there are two distinct equilibria on the boundaries of $\mathcal{M}$ but, just outside $\mathcal{M}$, there is a unique equilibrium. Hence, as one enters $\mathcal{M}$, at least one "new" equilibrium must appear. This is what is demonstrated by the third and fourth points of the proposition: a strictly lower equilibrium, Ω_L, appears as one enters $\mathcal{M}$ from below, and a strictly higher equilibrium, Ω_H, appears as one enters $\mathcal{M}$ from above.

V.1 Public Information and Total Knowledge

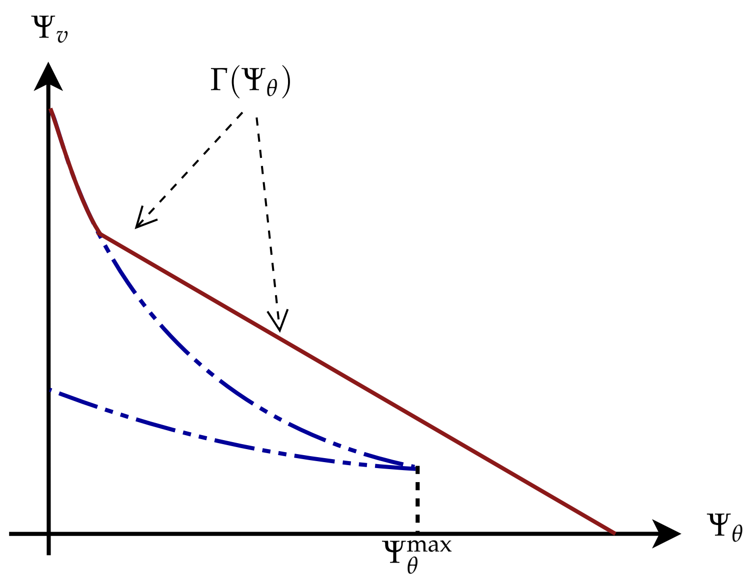

Figure 3: The solid line is the boundary $\Psi_v = \Gamma(\Psi_\theta)$ above which welfare decreases in (Ψ_θ, Ψ_v). The dotted line shows the boundaries of the set $\mathcal{M}$.

We now explain the impact of increasing public information on workers' posterior precision about v:

$$\psi_v + \psi_\theta \Omega^2 + \Psi_v + \Psi_\theta \Omega^2,$$

what we call "total knowledge."

Proposition V.3 (Knowledge in the (Ψ_θ, Ψ_v) plane)**.** *Suppose that households coordinate on the highest knowledge equilibrium and fix some (ψ_θ, ψ_v). Let, for $(\Delta_0, \Delta_1) \in \mathbb{R}_+^2$,*

$$\begin{aligned}\Gamma(\Psi_\theta) &= \max\{0, \Delta_0 - \Delta_1 \Psi_\theta, U(\Psi_\theta)\} \text{ whenever } U(\Psi_\theta) \text{ is defined} \\ &= \max\{0, \Delta_0 - \Delta_1 \Psi_\theta\} \qquad \text{otherwise,}\end{aligned}$$

Then, there exists $(\Delta_0, \Delta_1) \in \mathbb{R}_+^2$ such that, when $\Psi_v \neq U(\Psi_\theta)$, workers' posterior precision about v decreases continuously in (Ψ_θ, Ψ_v) if and only if $\Psi_v < \Gamma(\Psi_\theta)$. When, $\Psi_v = U(\Psi_\theta)$, welfare jumps down.

The boundary $\Gamma(\Psi_\theta)$ is shown in Figure 3. Along any increasing curve in the (Ψ_θ, Ψ_v) plane that passes through the origin, total knowledge will have a U shape: it will decrease first, reach a minimum when crossing the boundary $\Psi_v = \Gamma(\Psi_\theta)$ from below, and increase thereafter. Note also that the curve may cross the boundary $\Psi_v = U(\Psi_\theta)$ before or at the same time as the boundary $\Psi_v = \Gamma(\Psi_\theta)$. At that crossing point, the high equilibrium disappears and total knowledge will have a negative jump.

One may suspect that total knowledge can decrease only when there are multiple equilibria coupled with that fact that we have arbitrarily chosen to focus on the highest knowledge equilibrium. The proposition clarifies that it is not the case: for example, below the set $\mathcal{M}$, the equilibrium is unique and total knowledge decreases.

VI Proofs

VI.1 Proof of Propositions V.1 and V.2

Recall that the fixed point equation is:

$$\Omega = \frac{1}{(1+\delta)\gamma} + \frac{\delta}{1+\delta} \frac{\psi_v + \Omega^2 \psi_\theta}{\Psi_v + \Omega^2 \Psi_\theta + \psi_v + \Omega^2 \psi_\theta},$$

where $\psi_v = \psi_a/(\gamma\phi)$. We start making a change of variable:

$$A \equiv \frac{\Omega(1+\delta)}{\delta} - \frac{1}{\delta\gamma} \text{ and we let } \kappa \equiv \frac{1}{\delta\gamma}.$$

Note that, while Ω belongs to the interval $(\frac{1}{\gamma(1+\delta)}, \frac{1+\delta\gamma}{\gamma(1+\delta)})$, A belongs to the interval $(0,1)$. In terms of our newly defined variable, A, the fixed point equation becomes:

$$A = \frac{\psi_v + \left(\frac{\delta}{1+\delta}\right)^2 (A+\kappa)^2\psi_\theta}{\Psi_v + \left(\frac{\delta}{1+\delta}\right)^2 (A+\kappa)^2\Psi_\theta + \psi_v + \left(\frac{\delta}{1+\delta}\right)^2 (A+\kappa)^2\psi_\theta}$$

Next, we let:

$$\lambda \equiv \left(\frac{\delta}{1+\delta}\right)^2 \frac{\psi_\theta}{\psi_v},\ \rho_v \equiv \frac{\Psi_v}{\psi_v}, \text{ and } \rho_\theta \equiv \frac{\Psi_\theta}{\psi_\theta},$$

so that the fixed point equation becomes:

$$A = \frac{1+\lambda(A+\kappa)^2}{\rho_v + \lambda\rho_\theta(A+\kappa)^2 + 1 + \lambda(A+\kappa)^2} \tag{VI.9}$$

This equation is equivalent to:

$$\begin{aligned}\rho_v = G(A,\rho_\theta) &\equiv -(1+\rho_\theta)\lambda(A+\kappa)^2 + \frac{\lambda}{A}(A+\kappa)^2 + \frac{1}{A} - 1 \\ &= -(1-2\kappa\lambda+\kappa^2\lambda(1+\rho_\theta)) + \frac{1+\kappa^2\lambda}{A} + \lambda(1-2\kappa(1+\rho_\theta))A - \lambda(1+\rho_\theta)A^2.\end{aligned}$$

Then, clearly:

(R1) $G(A,\rho_\theta)$ goes to infinity as A goes to zero.

(R2) $G(1,\rho_\theta) = -(1+\kappa)^2\lambda\rho_\theta < 0$.

Taking derivatives we have find that $\partial G/\partial A = D(A,\rho_\theta)/A^2$ where:

$$\begin{aligned} D(A,\rho_\theta) &= -1 - \kappa^2\lambda + (\lambda - 2\kappa\lambda(1+\rho_\theta))A^2 - 2\lambda(1+\rho_\theta)A^3 \\ \frac{\partial D(A,\rho_\theta)}{\partial A} &= 2\lambda A\left[1 - 2\kappa(1+\rho_\theta) - 3A(1+\rho_\theta)\right]\end{aligned}$$

Note as well that $D(0,\rho_\theta) = -(1+\kappa^2\lambda) < 0$ and $D(1,\rho_\theta) = -1-(1+\kappa)\lambda(1+\kappa+2\rho_\theta) < 0$. If $1-2\kappa(1+\rho_\theta) > 0$, then $D(A,\rho_\theta)$ is a hump-shaped function of A, with a unique maximum at $A = \frac{1-2\kappa(1+\rho_\theta)}{3(1+\rho_\theta)} < 1$. If $1-2\kappa(1+\rho_\theta) \leq 0$, then D is strictly decreasing, and thus the maximum is $D(0,\rho_\theta)$. Then we have that:

$$\max_{A\in[0,1]} D(A,\rho_\theta) = \begin{cases} -(1+\lambda\kappa^2) + \dfrac{\lambda\left(1-2\kappa(1+\rho_\theta)\right)^3}{27(1+\rho_\theta)^2} & ;\text{if } 1-2\kappa(1+\rho_\theta) > 0. \\ -(1+\lambda\kappa^2) & ;\text{if } 1-2\kappa(1+\rho_\theta) \leq 0 \end{cases}$$

If follows that the function $G(A,\rho_\theta)$ is strictly decreasing in $A \in [0,1]$ if and only if the maximum of $D(A,\rho_\theta)$ is negative, or if and only if

$$F(\rho_\theta) \equiv \frac{\lambda(1-2\kappa(1+\rho_\theta))^3}{27(1+\rho_\theta)^2} < 1+\lambda\kappa^2. \tag{VI.10}$$

One easily sees that, if the inequality holds for $\rho_\theta = 0$, then it also holds for all ρ_θ. Indeed, if $F(0) < 0$, then $F(\rho_\theta) < 0$ for all $\rho_\theta > 0$, and the inequality holds for all $\rho_\theta > 0$. If $F(0) > 0$, then $F(\rho_\theta)$ is decreasing when it is positive, so it achieves its maximum at $\rho_\theta = 0$. Summing up, we find that the maximum of $D(A,\rho_\theta)$ is negative for all ρ_θ if

$$F(0) \leq 1+\lambda\kappa^2 \Leftrightarrow \lambda(1-2\kappa)^3 \leq 27(1+\lambda\kappa^2) \Leftrightarrow \lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] \leq 27$$

Otherwise, if this condition is not satisfied, we have the following situation: $F(0) > 1+\lambda\kappa^2$, $F(\rho_\theta)$ is strictly decreasing when it is positive, and $F(\rho_\theta) < 1+\lambda\kappa^2$ for ρ_θ large enough. It follows that there exists a unique $\bar{\rho}_\theta$ such that $F(\rho_\theta) < 1+\lambda\kappa^2$ if and only if $\rho_\theta > \bar{\rho}_\theta$. Wrapping up, we can say:

(R3) If $\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] \leq 27$ or $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta \geq \bar{\rho}_\theta\right)$, then $G(A,\rho_\theta)$ is strictly decreasing in $A \in (0,1)$.

(R4) If $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta < \bar{\rho}_\theta\right)$, then there exist two functions $a_1(\rho_\theta)$ and $a_2(\rho_\theta)$ such that:

$$0 < a_1(\rho_\theta) < \frac{1-2\kappa(1+\rho_\theta)}{3(1+\rho_\theta)} < a_2(\rho_\theta) < 1$$

and $G(A,\rho_\theta)$ is decreasing in $A \in (0, a_1(\rho_\theta))$, increasing in $A \in (a_1(\rho_\theta), a_2(\rho_\theta))$, and decreasing again in $A \in (a_2(\rho_\theta), 1)$.

(R5) The function $a_1(\rho_\theta)$ is increasing in ρ_θ while $a_2(\rho_\theta)$ is decreasing.

This follows from noticing that an increase in ρ_θ reduces $D(A,\rho_\theta)$ for all $A \in (0,1)$.

The hump-shape of $D(A, \rho_\theta)$ together with the fact that $a_1(\rho_\theta)$ and $a_2(\rho_\theta)$ are zeros of $D(A, \rho_\theta)$ delivers the result.

(R6) As ρ_θ approaches $\bar{\rho}_\theta$, both $a_1(\rho_\theta)$ and $a_2(\rho_\theta)$ tend to $\frac{1-2\kappa(1+\rho_\theta)}{3(1+\rho_\theta)}$.

Indeed consider a sequence of ρ_θ converging to ρ_θ. The corresponding sequence of $a_1(\rho_\theta)$ is increasing and bounded above (below), so it must converge as $\rho_\theta \to \bar{\rho}_\theta$. Let the limit be $\bar{a}_1$. By continuity given that $a_1(\rho_\theta)$ is a zero of $D(A, \rho_\theta)$, it follows that $D(\bar{a}_1, \bar{\rho}_\theta) = 0$. But the result follows because, when $\rho_\theta = \bar{\rho}_\theta$ the maximum of $D(A, \rho_\theta)$ is zero and is achieved uniquely at $A = \frac{1-2\kappa(1+\rho_\theta)}{3(1+\rho_\theta)}$. A similar argument can be applied to determine the limit of $a_2(\rho_\theta)$ as $\rho_\theta \to \bar{\rho}_\theta$.

Now we are ready to describe all the solutions of the equation $G(A, \rho_\theta) = \rho_v$. Note that (R1) and (R2) implies that there is at least one solution in the interval $(0, 1)$. In case (R3), the function $G(A, \rho_\theta)$ is strictly decreasing in A so that solution is unique. In the other case, (R4), we start by defining $B^{(1)}(\rho_\theta) \equiv G(a_1(\rho_\theta), \rho_\theta)$ and $B^{(2)}(\rho_\theta) \equiv G(a_2(\rho_\theta), \rho_\theta)$. We have the following:

(R7) The function $B^{(i)}(\rho_\theta)$ is strictly decreasing in ρ_θ for $i \in \{1, 2\}$.

The proof of (R7) follows by noticing that by construction $\left.\frac{\partial G(A,\rho_\theta)}{\partial A}\right|_{A=a_i} = 0$. So it follows that:

$$\frac{dB^{(i)}}{d\rho_\theta} = \left.\frac{\partial G^{(i)}(A, \rho_\theta)}{\partial \rho_\theta}\right|_{A=a_i} = -(a_i(\rho_\theta) + \kappa)^2 \lambda < 0. \tag{VI.11}$$

Now the set of equilibria can be described graphically, as follows. Figure 4 shows a plot of $G(A, \rho_\theta)$ as a function of A. The local minimum is achieved at $A = a_1(\rho_\theta)$ and is equal to $B^{(1)}(\rho_\theta)$, and the local maximum is achieved at $a_2(\rho_\theta)$ and is equal to $B^{(2)}(\rho_\theta)$. The figure makes it clear that

- if $\rho_v < B^{(1)}(\rho_\theta)$, then there is a unique equilibrium, which is greater than $a_2(\rho_\theta)$;
- if $\rho_v = B^{(1)}(\rho_\theta)$, then there are two equilibria, $A_L = a_1(\rho_\theta) < a_2(\rho_\theta) < A_H$;
- if $B^{(1)}(\rho_\theta) < \rho_v < B^{(2)}(\rho_\theta)$, then there are three equilibria $A_L < a_1(\rho_\theta) < A_M < a_2(\rho_\theta) < A_H$;
- if $\rho_v = B^{(2)}(\rho_\theta)$, then there are two equilibria, $A_L < a_1(\rho_\theta) < a_2(\rho_\theta) = A_H$;
- if $\rho_v > B^{(2)}(\rho_\theta)$, then there is a unique equilibrium, which is smaller than $a_1(\rho_\theta)$.

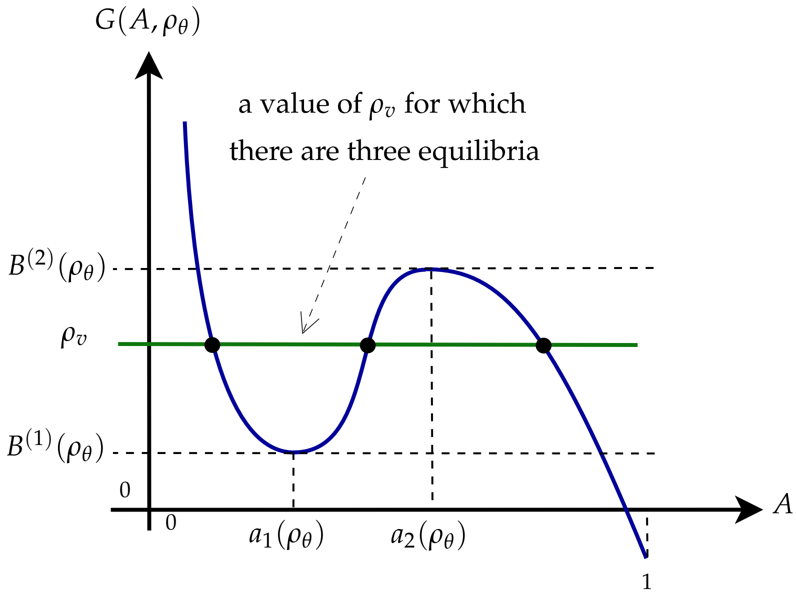

Figure 4: The function $G(A, \rho_\theta)$.

The equilibrium A then maps into the equilibrium Ω according to the equation $\Omega = \frac{\delta}{1+\delta}\left(A + \frac{1}{\gamma\delta}\right)$. Replacing $\rho_\theta = \Psi_\theta/\psi_\theta$, $\rho_v = \Psi_v/\psi_v$, we find that the set $\mathcal{M}$ is described parametrically by:

$$0 \leq \Psi_\theta < \Psi_\theta^{\max}$$
$$\text{and} \quad L(\Psi_\theta) \leq \Psi_v \leq U(\Psi_\theta),$$

where:

$$\begin{aligned}
\Psi_\theta^{\max} &\equiv \bar{\rho}_\theta \psi_\theta \\
L(\Psi_\theta) &= \psi_v B^{(1)}\left(\frac{\Psi_\theta}{\psi_\theta}\right) \\
U(\Psi_\theta) &= \psi_v B^{(2)}\left(\frac{\Psi_\theta}{\psi_\theta}\right).
\end{aligned}$$

Note that $L(\Psi_\theta)$ and $U(\Psi_\theta)$ are implicit functions of (ψ_θ, ψ_v). By (R7), the boundaries $L(\Psi_\theta)$ and $U(\Psi_\theta)$ are decreasing functions of Ψ_θ. Also, because $a_2(\rho_\theta)$ is decreasing, it follows from (VI.11) that $B^{(2)}(\rho_\theta)$ is convex. Lastly, by (R6), we have that $L(\Psi_\theta^{\max}) = U(\Psi_\theta^{\max})$.

Now the first two points of Proposition V.2 follow from (R8)-(R12). The last two points follow from an application of the IFT on the boundaries. Namely, on the upper boundary, one applies the IFT at $A = A_L$, because $\partial G/\partial A(A_L) < 0$. This shows that the unique equilibrium above the boundary converges towards A_L as (Ψ_θ, Ψ_v) approaches the boundary from above. Note that the reasoning does not apply to the other equilibrium, A_H: indeed, because $\partial G/\partial A(A_H) = 0$, one cannot apply the IFT at $A = A_H$. Similarly, on the lower boundary, one applies the IFT to $A = A_H$, but cannot apply it at $A = A_L$ because $\partial G/\partial(A_L) = 0$.

VI.2 Proof of Proposition V.3

We keep all the notations of the previous proof and let $A_\star$ denotes the largest fixed point of equation (VI.9), corresponding to the highest-knowledge equilibrium. Using this equation, the household's posterior precision about v can be written:

$$\Pi = \psi_v \left(\rho_v + \rho_\theta \lambda \left(A_\star + \kappa\right)^2 + 1 + \lambda \left(A_\star + \kappa\right)^2 \right).$$

Using the fixed-point equation, we find that:

$$\frac{\Pi}{\psi_v} = \frac{1}{A_\star} + \frac{\lambda(A_\star + \kappa)^2}{A_\star}.$$

Taking derivatives with respect to $A_\star$, one finds that

$$\frac{\partial \Pi}{\partial A_\star} = \frac{-(1 + \lambda\kappa^2) + \lambda A_\star^2}{A_\star^2},$$

Now recall that $A_\star$ is decreasing in both Ψ_θ and Ψ_v. Given our description of the equilibrium set, along an increasing path of Ψ_θ and Ψ_v, $A_\star$ is continuous except at the upper frontier where it jumps down. It then follows that,

(R13) If $A_\star$ is continuous, then a marginal increase in public information reduces total knowledge if and only if $A_\star > \alpha(\lambda, \kappa) \equiv \sqrt{1/\lambda + \kappa^2}$.

In what follows, to simplify notations, we suppress the explicit dependence of $\alpha(\lambda, \kappa)$ on (λ, κ). Now we show:

(R14) Suppose there is a unique equilibrium. Then, a marginal increase in public information decreases total knowledge if and only if

$$\rho_v < C_0 - C_1 \rho_\theta \equiv C(\rho_\theta),$$

where $C_0 = \left[1 + \lambda \left(\alpha + \kappa\right)^2\right] \left[1/\alpha - 1\right]$ and $C_1 = \lambda \left(\alpha + \kappa\right)^2$.

Indeed, recall that if there is a unique equilibrium, $A_\star$ is continuous, and the best reply is above the 45-degree line for all $A < A_\star$ and below the 45-degree line for all $A > A_\star$. Thus, $\alpha < A_\star$ if and only if, when $A = \alpha$, the best reply is above the 45-degree line. Plugging $A = \alpha$ in the fixed point equation and rearranging, one obtains the condition (R14).

Now consider parameters λ and ρ_θ such that there are multiple equilibria. We start by showing:

(R15) Suppose $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta < \bar{\rho}_\theta\right)$. Then, $a_1(\rho_\theta) > \alpha$.

Indeed, note that $D(A, \rho_\theta) = -\lambda\alpha^2 + \lambda A^2 +$ negative terms. Therefore, if $A \leq \alpha$, then $D(A, \rho_\theta) < 0$. Since $D(A, \rho_\theta)$ is negative for $A < a_1(\rho_\theta)$ and positive for $a_1(\rho_\theta) < A < a_2(\rho_\theta)$, it thus follows that $\alpha < a_1(\rho_\theta)$. Next, we prove:

(R16) Suppose $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta < \bar{\rho}_\theta\right)$. Then, for all $\rho_v < B^{(2)}(\rho_\theta)$, total knowledge decreases continuously with public information.

Indeed, first recall from (R8) that for all $\rho_v < B^{(1)}(\rho_\theta)$, there is a unique equilibrium, $A_\star$, that is larger than $a_2(\rho_\theta)$, and thus larger than $a_1(\rho_\theta)$. Next, for all $B^{(1)}(\rho_\theta) \leq \rho_v < B^{(2)}(\rho_\theta)$, the largest equilibrium, $A_\star$, is also larger than $a_1(\rho_\theta)$. Because (R15) showed that $a_1(\rho_\theta) > \alpha$, it follows that for all $\rho_v < B^{(2)}(\rho_\theta)$, $A_\star > \alpha$. Thus, from (R13), we know that, for all $\rho_v < B^{(2)}(\rho_\theta)$, a marginal increase in public information reduces total knowledge.

Taken together, (R14) and (R16), we can generalize to:

(R17) Total knowledge decreases continuously with public information if and only if

$$\begin{aligned} \rho_v &< \max\{B^{(2)}(\rho_\theta), C(\rho_\theta)\} && \text{whenever } B^{(2)}(\rho_\theta) \text{ is defined} \\ \rho_v &< C(\rho_\theta) && \text{otherwise.} \end{aligned}$$

total knowledge jumps down with public information if and only if $\rho_v = B^{(2)}(\rho_\theta)$.

Indeed if $B^{(2)}(\rho_\theta)$ is not defined then this is the same as (R14). If $B^{(2)}(\rho_\theta)$ is defined, let's start with the "only if" part. If public information continuously decreases total knowledge, then either (i) $\rho_v < B^{(2)}(\rho_\theta)$ or (ii) $\rho_v > B^{(2)}(\rho_\theta)$, the equilibrium is unique and by (R14), $\rho_v < C(\rho_\theta)$. Either way, $\rho_v < \max\{B^{(2)}(\rho_\theta), C(\rho_\theta)\}$. For the "if" part, suppose that $\rho_v < \max\{B^{(2)}(\rho_\theta), C(\rho_\theta)\}$. Then, either $\rho_\theta \leq B^{(2)}(\rho_\theta)$ and by (R17) a marginal increase in public information decreases total knowledge. Or $\rho_v > B^{(2)}(\rho_\theta)$

and $\rho_v < C(\rho_\theta)$, the equilibrium is unique, and a marginal increase in public information decreases total knowledge. The last point follows directly from the fact that, when $\rho_v = B^{(2)}(\rho_\theta)$, the aggregate weight $A_\star$ jumps down.

The Proposition follows by setting

$$\Delta_0 = \psi_\theta C_0$$
$$\Delta_1 = \psi_\theta C_1.$$

An Additional Result for Drawing Figure 3

We now characterize further the shape of the boundary. Namely, we show

Lemma VI.1. *Suppose* $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta < \bar{\rho}_\theta\right)$. *Then, there exists* $\hat{\rho}_\theta \in [0, \bar{\rho}_\theta]$ *such that* $B^{(2)}(\rho_\theta) > C(\rho_\theta)$ *if and only if* $\rho_\theta < \hat{\rho}_\theta$.

In words, the lemma says that $B^{(2)}(\rho_\theta)$ is above $C(\rho_\theta)$ when ρ_θ is less than $\hat{\rho}_\theta$ and below $C(\rho_\theta)$ thereafter. So the boundary of result (R17) coincides with $B^{(2)}(\rho_\theta)$ for low values of ρ_θ, and with $C(\rho_\theta)$ for high values. Also, it can be that $\hat{\rho}_\theta = 0$; in that case, $C(\rho_\theta) > B^{(2)}(\rho_\theta)$ all along.

We now proceed to prove the lemma, starting with

(R18) Suppose $\left(\lambda\left[(1-2\kappa)^3 - 27\kappa^2\right] > 27 \text{ and } \rho_\theta < \bar{\rho}_\theta\right)$. Then, $B^{(2)}(\rho_\theta) < C(\rho_\theta)$ if and only if, when $\rho_v = C(\rho_\theta)$, there is a unique equilibrium.

Consider first the "only if" part. Suppose that $B^{(2)}(\rho_\theta) < C(\rho_\theta)$ and let $\rho_v = C(\rho_\theta)$. Then, $\rho_v > B^{(2)}(\rho_\theta)$ so by (R12) there is a unique equilibrium. We proceed with the "if" part. Suppose that, when $\rho_v = C(\rho_\theta)$, there is a unique equilibrium. Thus, (ρ_θ, ρ_v) could be either above the upper boundary, i.e., $\rho_v > B^{(2)}(\rho_\theta)$, or below the lower boundary, i.e., $\rho_v < B^{(1)}(\rho_\theta)$. But, if (ρ_θ, ρ_v) were below the lower boundary, then, by (R16), a marginal increase in ρ_θ would result in a marginal decrease total knowledge, which is impossible because $\rho_v = C(\rho_\theta)$. Next, we show:

(R19) Let $\rho_v = C(\rho_\theta)$. Then, there is more than one equilibrium if and only if $\rho_\theta \leq \hat{\rho}_\theta \leq \bar{\rho}_\theta$, where $\hat{\rho}_\theta \equiv \max\left\{0, \max_{A\geq 0}\left\{\frac{A-\alpha}{A(A+\alpha+2\kappa)}\right\} - 1\right\}$.

Indeed, consider the equilibrium equation $G(A, \rho_\theta) = \rho_v$, when $\rho_v = C(\rho_\theta)$. By construction, when $\rho_v = C(\rho_\theta)$, $A = \alpha$ solves the fixed point equation. Moreover, straightforward algebraic manipulations of the fixed point equation shows that

$$\begin{aligned}
&G(A,\rho_\theta) = C(\rho_\theta)\\
\Leftrightarrow &A\left[\frac{1+\lambda(\alpha+\kappa)^2}{\alpha} + \lambda(1+\rho_\theta)\left((A+\kappa)^2-(\alpha+\kappa)^2\right)\right] = 1+\lambda(A+\kappa)^2\\
\Leftrightarrow &\left[A-\alpha+\lambda A(\alpha+\kappa)^2-\lambda\alpha(A+\kappa)^2\right] + \lambda A\alpha(1+\rho_\theta)\left((A+\kappa)^2-(\alpha+\kappa)^2\right) = 0\\
\Leftrightarrow &\left[A-\alpha+\lambda\left(A\alpha^2+A\kappa^2-\alpha A^2-\alpha\kappa^2\right)\right] + \lambda A\alpha(1+\rho_\theta)(A-\alpha)(A+\alpha+2\kappa) = 0\\
\Leftrightarrow &(A-\alpha)\left(1+\lambda\kappa^2-\lambda\alpha A\right) + \lambda A\alpha(1+\rho_\theta)(A-\alpha)(A+\alpha+2\kappa) = 0\\
\Leftrightarrow &(A-\alpha)\,\lambda\alpha\,(\alpha-A) + \lambda A\alpha(1+\rho_\theta)(A-\alpha)(A+\alpha+2\kappa) = 0\\
\Leftrightarrow &\lambda\alpha\,(A-\alpha)\left[\alpha-A+A(1+\rho_\theta)(A+\alpha+2\kappa)\right] = 0,
\end{aligned}$$

where we used $\lambda\alpha^2 = 1+\lambda\kappa^2$ in order to move from the third-to-last to the second-to-last line. Note that the term in square bracket is strictly positive when $A \leq \alpha$. Therefore, we have more than one equilibrium if and only if the term in square bracket has a zero. Or, if and only if there exists $A > \alpha$ such that

$$1+\rho_\theta = \frac{A-\alpha}{A(A+\alpha+2\kappa)}.$$

Clearly, this equation has a solution if and only if $1+\rho_\theta$ is less than the maximum of the right-hand side. Note that, if there exists some ρ_v such that there is more than one equilibrium, then we already know from (R4) that $\rho_\theta \leq \bar{\rho}_\theta$. Thus, it follows that $\hat{\rho}_\theta \leq \bar{\rho}_\theta$.

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