Reputation and Sovereign Default*

Manuel Amador† Christopher Phelan‡

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Abstract

This paper presents a continuous-time model of sovereign debt. In it, a relatively impatient sovereign government's hidden type switches back and forth between a commitment type, which cannot default, and an optimizing type, which can; outside lenders have particular beliefs regarding how a commitment type should borrow for any given level of debt and bond price. In any Markov equilibrium, the optimizing type mimics the commitment type when borrowing, revealing its type only by defaulting on its debt at random times. The equilibrium features a "graduation date": a finite amount of time since the last default, after which time reputation reaches its highest level and is unaffected by not defaulting. Before such date, not defaulting always increases the country's reputation. For countries that have recently defaulted, bond prices and the total amount of debt are increasing functions of the amount of time since the country's last default. For countries that have not recently defaulted (i.e., those that have graduated), bond prices are constant.

1 Introduction

This paper presents a continuous-time model of sovereign debt where a sovereign government's reputation evolves over time. In the model, a relatively impatient sovereign government's hidden type exogenously switches back and forth between a *commitment* type, which cannot default, and an *optimizing* type, which can default on the country's debt at any time. We consider the government's reputation at any time to be the international lending markets' Bayesian posterior that the country's government is the commitment (or trustworthy) type.[1]

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†University of Minnesota, Federal Reserve Bank of Minneapolis, and NBER.

‡University of Minnesota, Federal Reserve Bank of Minneapolis, and NBER.

[1] Cole, Dow and English (1995), Alfaro and Kanczuk (2005), and D'Erasmo (2011) also feature alternating government types in a sovereign debt model. Our paper differs from Cole et al. (1995) and Alfaro and Kanczuk (2005) by focusing exclusively on the behavior of potentially defaulting government types. It differs from D'Erasmo (2011) in that we impose no exogenous cost of default.

We show that such a model helps to explain several important characteristics of actual sovereign debt: First, some countries are considered "serial defaulters." In particular, countries that have recently defaulted are considered more likely to default again and pay correspondingly higher interest rates on their debt. Second, countries that have recently defaulted can sustain much less debt relative to their GDP than countries that have not recently defaulted — a phenomenon referred to as "debt intolerance." Third, some countries do eventually "graduate" into the set of "debt-tolerant" or relatively trusted countries, but only after decades of good behavior.[2] Fourth, default events and variations in interest rate spreads are weakly associated with fundamentals such as debt and output ratios; they cannot be precisely predicted.[3] Finally, countries with strictly *positive* trade deficits do sometimes default, a fact difficult to reconcile with standard sovereign debt models.[4] Our model captures each of these characteristics.

We present a Markov equilibrium for our model where a borrower country's debt and the price of its bonds are both strictly increasing functions of the amount of time since its last default (and a country can issue new bonds *immediately* after a default, albeit at high interest rates). In fact, the highest-priced bonds are those issued by the countries with the *most* debt (the countries that have experienced the longest amount of time since a default). This occurs because, from the perspective of a lender, the probability of default is a strictly decreasing function of the amount of time since the last default. Thus, our equilibrium displays both debt intolerance (high interest rates paid by countries with low debt levels) and serial default (a relatively high probability of default by countries that have recently defaulted). Finally, our equilibrium features a "graduation date" — an amount of time since the last default T such that if a government goes T amount of time without defaulting, foreign lenders become certain it is the commitment type (and offer the lowest interest rate on its debt). At this point, the reputation of the country and the price of its bonds never change as long as default does not occur.

Games where informed players (in our case, a government who knows its type) have rich action spaces are notoriously difficult to characterize. We make progress by making the default decision of the opportunistic type the only strategic choice. That is, we assume foreign lenders have *particular* beliefs regarding how a commitment type *should* borrow for any given level of debt and bond price (and believe that the government must be the optimizing type if it deviates from this borrowing behavior). Given this assumption, the *commitment type* plays a passive role,

[2] Reinhart, Rogoff and Savastano (2003) first coined the term "debt intolerance" and discuss the history of serial defaulters and graduation.

[3] See Tomz and Wright (2007) and Aguiar and Amador (2014). This fact has led some researchers to postulate the need for self-fulfilling debt crisis as a necessary ingredient in models of sovereign debt, as in Aguiar, Chatterjee, Cole and Stangebye (2016).

[4] In the Eaton-Gersovitz model studied by the quantitative literature, if a country is able to generate a positive external trade balance for some level of debt issuances, then it will not default. See Arellano (2008), Proposition 2, which, although stated for iid shocks, holds more generally.

following its expected behavior.

The *optimizing type*, on the other hand, faces at all times the strategic decision of whether to mimic the commitment type or *default*, which we model, as is usual in the sovereign debt literature, as wiping out all of its existing debt. The benefit of defaulting is, of course, that the country makes no further coupon payments on its debt. The cost is that outside lenders, at the time of default, become certain the country is the optimizing type. (Unlike much of the quantitative sovereign debt literature, we assume no *direct* costs of default.)[5]

We first show for any *particular* specification of outside lender expectations regarding how a commitment type should act, how to solve for a Markov equilibrium as the solution to a pair of ordinary differential equations. We next show that as long as these expectations are in a reasonable class (namely, the country borrows less if it has more debt, borrows more if it faces better bond prices, and borrows a strictly positive amount if it has no debt and a good enough bond price), *all such equilibria look qualitatively identical* — the exact specification of lender expectations does not qualitatively matter.

We show that in our Markov equilibrium, for an endogenous finite period of time T after a default, an optimizing government sets a positive but finite hazard rate of defaulting which depends only on how much time has occurred since the last default. Further, as this amount of time since the last default approaches T, this arrival rate of default, *conditional on the government being the opportunistic type,* approaches infinity (certain immediate default).

Nevertheless, from the perspective of a foreign lender, the probability of default is *decreasing* in the amount of time since the last default. This happens because although an optimizing government is *more* likely to default the longer it has been since the last default, whether the country's government actually *is* the optimizing type is *less* likely the longer it has been since a default, and this latter effect dominates. This implies that if the amount of time since the last default is positive but less than T, the country has an interior reputation that increases over time but after T is certainly the commitment type.

A country's reputation increases in the amount of time since its last default because the commitment type defaults with zero probability and the optimizing type defaults not only with positive probability but with a probability high enough to counter any drift in reputation due to exogenous type switching.

But this mixing imposes strict requirements on the path of play. In particular, we show this willingness to mix implies that the optimizing type must receive constant *net* payments from foreign lenders. We show that the optimizing types never, on net, repay. But that foreign lenders

[5]The presence of the commitment type allows us to avoid the Bulow and Rogoff (1989) result of no debt in equilibrium absent direct default costs. For an alternative analysis of reputation, based on trigger strategies, see Kletzer and Wright (2000) and Wright (2002).

always break even in expectation then implies that future commitment types must be the ones paying back the debt. That is, in equilibrium, optimizing types extract constant rents from future commitment types. Only a commitment type will run a trade surplus or a trade deficit smaller than the one extracted in equilibrium by the optimizing type. In fact, optimizing types default with probability one once they cannot extract this constant rent. They do not wait for the trade deficit to actually become negative before defaulting.[6]

These characterizations are derived assuming the borrower country starts the game with zero debt and zero reputation (outside lenders are convinced it is the optimizing type). This is the relevant subgame after the first default. In this subgame there is, in equilibrium, a specific debt level associated with each reputation, which we call the country's *appropriate* debt level. We next consider starting the game with arbitrary values of reputation and debt. We show that if the country's debt is above the level appropriate for its initial reputation, the equilibrium calls for immediate probabilistic default. If the country defaults, the game reverts to the subgame with zero debt and zero reputation. If the country does not default, its debt stays the same, but the country's reputation (from the fact that it did not default when it was supposed to with positive probability) jumps to its appropriate level. Next, we show that if the country's debt starts below its appropriate level, the equilibrium calls for the opportunistic government to set the probability of default to zero for a finite amount of time until its reputation and debt converge to the appropriate levels.

This characterization allows us to then consider unanticipated shocks, and we show an asymmetric reaction. If a *good* shock occurs (say, a permanent increase in the borrower country's endowment), this makes its debt inappropriately low relative to its reputation, and our model predicts a zero chance of default for some finite amount of time (followed by the same dynamics as when starting from no debt and no reputation). Alternatively, if a *bad* shock occurs, this makes the country's debt inappropriately high relative to its reputation, and our model predicts immediate probabilistic default (with the probability of default an increasing function of the size of the shock). As in our baseline model where defaults are not perfectly predictable, here the realized *reaction* of a country to a bad shock is also not perfectly predictable from its debt-output ratio, as in the data.

The model in this paper shares several features with the taxation paper of Phelan (2006). In that paper, a government can be either a commitment type, which must tax output at a low rate, or an optimizing type, which taxes either the low rate or confiscates all output, with exogenous hidden type switches as in the model presented here. Like this paper, the optimizing type mixes

[6] In a recent paper, Aguiar, Chatterjee, Cole and Stangebye (2017) exploit equilibrium mixing where the government is indifferent between default or not, to generate dynamics of debt and spreads that more closely match the data, improving the fit of benchmark quantitative Eaton and Gersovitz (1981) sovereign debt model.

for some time and then separates from the commitment type in the Markov equilibrium. Apart from these two characteristics, however, the implications of the two models differ. In this paper, we explicitly model the relation between *debt*, a payoff-relevant state variable, and reputation, and our characterization relies heavily on the specific features of debt contracts. Contrary to taxing, selling debt has an explicit time dimension: it is always an ex-post bad thing. This time dimension then drives our predictions for the paths of interest rates, debt levels, and default properties not present in earlier work. One striking difference is that in this paper, except off the equilibrium path, *there is no value to a good reputation* — the government's reputation affects government borrowing and the price of its debt, but not its continuation value.[7]

Section 2 presents our model, where we initially focus on the particular starting conditions of both debt and reputation having zero values. In Sections 3 and 4, we define and derive a Markov equilibrium of this game. In Section 5, we present a computed example. In Section 6, we show that the characteristics of the equilibrium derived in Section 4 hold for *all* Markov equilibria where the commitment type follows an expectation rule. In Section 7, we relax the assumption that the game starts with zero debt and zero reputation and characterize play of the game for all initial starting conditions for debt and reputation. Using these results, we are then able to show that the optimizing type will choose to reveal its type only by defaulting (and otherwise mimic the commitment type) and that the commitment type will follow the expectational rule if it is sufficiently impatient. Finally, in Section 7 we use our analysis of initial starting conditions to characterize reactions to unanticipated shocks. In Section 8, we present assumptions under which our equilibrium can be solved for in closed form and use these to perform comparative static exercises. Section 9 concludes.

2 The Environment

Time is continuous and infinite.

There is a small open economy whose government is endowed with a constant flow y of a consumption good.

There is a countable list of potential *governments* of the small open economy, with alternating types. With probability ρ_0, the first government on the list is the *commitment type* and with probability $(1 - \rho_0)$, the first government on the list is the *optimizing type*. The list then alternates between types. At any date $t \geq 0$, only one of the potential governments is in charge. With Poisson arrival rate ϵ, an optimizing type government is replaced by the next government on the

[7] Other recent models on reputation building include the discrete time models of Liu (2011) and Liu and Skrzypacz (2014), and the continuous time models of Faingold and Sannikov (2011), Board and Meyer-ter Vehn (2013), and Marinovic, Skrzypacz and Varas (2018).

list (a commitment type). With arrival rate δ, a commitment type government is replaced by an optimizing type. Such switches are private to the government.

We assume international financial markets are populated by a continuum of risk-neutral investors, who discount the future at rate $i > 0$.

The economy has, at any time t, an amount $b(t)$ of outstanding debt held by these risk-neutral investors. The bonds are of long duration and have a coupon that decays exponentially at rate λ, an exogenous parameter controlling the maturity of the bonds. We denote by f the first coupon of a newly issued bond, and thus $e^{-\lambda t} f$ represents the coupon t periods after issuance.[8] Without loss of generality, we set $f = i + \lambda$ to ensure that in equilibrium, the price of a bond is one if default cannot occur.

Initially, we assume that at time $t = 0$ there is no debt, $b_0 = 0$, and the government is known to be the optimizing type, that is, $\rho_0 = 0$. We will later consider the case where initial debt and initial reputation are not each set to zero. As time progresses, the government's debt, its reputation, and the price of its bonds will evolve as well.

2.1 Strategies

By assumption, a commitment type never defaults on its debts and will make any coupon payments that are due as long as it is in power. An optimizing type, however, can default. Once a default occurs, we assume that the current bond holders get no additional payments, and the stock of outstanding debt is set to zero. By defaulting, however, an optimizing government reveals its type and thus sets its reputation ρ (its probability of being the commitment type) to zero.

We further assume that the strategies are *Markov*: that continuation strategies are only a function of the level of debt $b(t)$ and the reputation $\rho(t)$. But then note that since both $b(t)$ and $\rho(t)$ are reset to zero (their initial values) upon default, making continuation strategies contingent only on $b(t)$ and $\rho(t)$ is the same as making continuation strategies contingent only on the amount of time since the last default (which we label τ) since $b(t)$ and $\rho(t)$ would themselves depend only on the amount of time since the last default. Essentially, the assumptions that $b_0 = \rho_0 = 0$ and Markov strategies, along with only optimizing types being able to default, ensure that default *restarts the game*. Thus, from here on we make strategies not a function of history or of debt or reputation, but simply a function of the time since the last default, τ.

For the commitment type, we assume that as long as it is in control, it follows a pre-specified expenditure rule determined by the expectations of international financial markets of how a commitment type should act. That is, as long as the commitment type is in control, the stock of debt

[8] We will later pay particular attention to the case where $\lambda = 0$, in which case bonds are consoles paying f at all future dates. In this case, along with several other assumptions, we solve the model analytically in closed form.

evolves according to

$$b'(\tau) = H(b(\tau), q(\tau)) \tag{1}$$

for some exogenous function H, where $q(\tau)$ represents the price of a bond τ periods after the last default. We will later find conditions on the payoffs for the commitment type and expectations of outside lenders such that the commitment type finds it optimal to follow H.

It follows from the sequential budget constraint that $c(\tau) = y - (i+\lambda)b(\tau) + q(\tau)(b'(\tau) + \lambda b(\tau))$, and thus consumption for the commitment type is determined and given by

$$C(b(\tau), q(\tau)) \equiv y - \underbrace{(i+\lambda)b(\tau)}_{\text{coupon payments}} + q(\tau)\underbrace{(H(b(\tau), q(\tau)) + \lambda b(\tau))}_{\text{bond sales}}.$$

We impose the following further conditions on $H(b, q)$ (and thus, implicitly, $C(b, q)$):

Assumption 1. *$H(b,q) : [0, \frac{y}{i+\lambda}] \times [0,1] \to \mathbb{R}$ is (i) continuous in both arguments, (ii) weakly decreasing in b, (iii) weakly increasing in q, and (iv) for all $(b,q) \in (0, \frac{y}{i+\lambda}) \times (0,1)$ such that $H(b,q) > 0$, differentiable in both arguments and strictly increasing in q.*

Assumption 2. *(boundedness). $H(0,0) \geq 0$, $H(\frac{y}{i+\lambda}, 1) \leq 0$ and there exists $\overline{H}$ such that $H(b,q) \leq \overline{H}$ for all $(b,q) \in [0, \frac{y}{i+\lambda}] \times [0,1]$.*

Assumption 3. *(impatience relative to outside lenders). $H(0, \frac{i+\lambda}{i+\lambda+\delta+\epsilon}) > 0$.*

Assumptions 1 and 2 together imply that if $b(0) = 0$, $b(\tau) \leq \frac{y}{i+\lambda}$ for all $\tau \geq 0$. This then ensures the commitment type's required coupon payment, $(i+\lambda)b(\tau)$, never exceeds y and thus it is *feasible* for the commitment type to never default regardless of the evolution of bond prices. Assumption 3 requires that outside lenders assume the commitment type is sufficiently impatient so as to be willing to borrow at the interest rate $i + \delta + \epsilon$ (where, again, the world interest rate is i). Note that these assumptions ensure $b(\tau)$ is continuous in time since the last default.

For the *optimizing* type, in addition to the Markov restriction, we impose for now a restriction that it always chooses a level of borrowing (and thus consumption) that is identical to that which would have been chosen by a commitment government facing the same debt and price. With this restriction, the only decision left under the control of the optimizing government is whether to default or not. We will later show that this restriction is without loss of generality: an optimizing government will have no incentive to reveal itself by choosing a level of borrowing or consumption different from the commitment government, without simultaneously defaulting on its debt.

Given this restriction, we assume that a strategy for an opportunistic government that has just taken power in period τ is a right-continuous and non-decreasing function $F_\tau : \mathbb{R}_+ \to [0,1]$, where $1 - F_\tau(s)$ defines the probability that this government does not default between period τ and $s \geq \tau$ inclusive, conditional on it remaining in power from τ to s. We let Γ denote the set of all such functions.

This formulation allows both jumps and smooth decreases in the survival probability, $1 - F_\tau$. If F_τ jumps up at $s \geq \tau$, this implies a strictly positive probability of defaulting at exactly date s. When F_τ smoothly increases, the probability of defaulting at exactly date s is zero. In this case, $F'_\tau(s)/(1-F_\tau(s))$ represents the hazard rate of default at that date (where $F'_\tau(s)$ represents the right derivative at s).

Our Markov restriction imposes that an opportunistic government that takes power in period τ follows a strategy that any previous opportunistic government would also follow from period τ onward if it were to remain in power up to period τ without defaulting (since both cases have the same debt and reputation). That is,

Definition 1. *A* **Markov strategy profile** *for opportunistic governments is a collection* $\{F_\tau\}_{\tau=0}^{\infty}$ *with* $F_\tau \in \Gamma$ *for all* τ*, such that*

$$1 - F_\tau(s) = (1 - F_\tau(m^-))(1 - F_m(s)) \text{ for all } 0 < \tau \leq m \leq s, \tag{2}$$

where $F_\tau(m^-) = \lim_{n \to m^-} F_\tau(n)$.

The above restriction implies that the function F_s is pinned down by F_0 for all s such that $F_0(s) < 1$. That is, if there is a strictly positive probability that the opportunistic government at time 0 reaches time s without defaulting, then it is possible to use the conditional probabilities inherent in F_0 to determine F_s. This however fails for s such that $F_0(s) = 1$, as in that case the opportunistic government at time 0 will not reach date s without defaulting first, and thus conditional probabilities are not defined. However, it is still necessary to define how an opportunistic government behaves at such dates, as there is a positive probability that an opportunistic government is in power at such a date due to two or more government type switches. Hence, F_0 is not in general sufficient to characterize opportunistic government play.

2.2 Payoffs

If the government does *not* default at period τ, it issues additional bonds $H(b(\tau), q(\tau)) + \lambda b(\tau)$ at endogenous price $q(\tau)$ and its consumption is $C(b(\tau), q(\tau))$. If the government *defaults*, then the game starts over (τ is reset to zero) with $b_0 = 0$. There are no direct costs of choosing to default and no restrictions on government borrowing from then on. In particular, it issues additional bonds $H(0, q(0))$ at endogenous price $q(0)$ and its consumption is $C(0, q(0))$.

The optimizing type receives a flow payoff equal to $u(c(\tau))$ as long as it is continuously in power, and discounts future payoffs at rate $r > 0$. We assume that $u : \mathbb{R}_+ \to [\underline{u}, \overline{u}]$ for some finite values $\underline{u}$ and $\overline{u}$, and that u is strictly increasing. We make no other assumptions on the preferences of the optimizing type. (A preview of our results is that our constructed Markov equilibrium is essentially *independent* of u and r. Other than more is preferred to less, and now is preferred to later; the preferences of the optimizing type will not matter at all.)

2.3 Beliefs

As noted, $\rho(\tau)$ represents the international market's beliefs that the government at period τ is the commitment type *after* it has not defaulted τ periods since the last default. We assume $\rho(\tau)$ is determined by Bayes' rule. (If the government defaults at period τ, Bayesian updating implies ρ and thus τ immediately jump to zero.)

By Bayes' rule, the probability that the commitment type is in government τ periods after the last default is

$$\rho(\tau) = \frac{\text{Probability of no default in } [0, \tau] \text{ and commitment type in power at } \tau}{\text{Probability of no default in } [0, \tau]}$$

As long as the probability of observing no default in $[0, \tau]$ is strictly positive, Bayes' rule pins down the market belief. Because the government type may switch in any positive interval, the only case where the denominator in the above equation is zero is when the opportunistic government defaults immediately at $\tau = 0$, that is when $F_0(0) = 1$ (a case that we handle below). It is helpful to write the evolution of ρ recursively in the following manner.

Consider first the case where $F_0(0) < 1$. Recall $F_\tau(\tau)$ is the probability that the opportunistic government, conditional on being in power τ periods since the last default, defaults at exactly period τ. Let $\rho(\tau^-) \equiv \lim_{n \to \tau^-} \rho(n)$ for $\tau > 0$ and $\rho(0^-) \equiv 0$. This represents the market belief an instant before the current default outcome.

If $F_\tau(\tau) > 0$, ρ jumps from $\rho(\tau^-)$ to

$$\rho(\tau) = \frac{\rho(\tau^-)}{\rho(\tau^-) + (1 - \rho(\tau^-))(1 - F_\tau(\tau))}. \tag{3}$$

Note this implies if $F_\tau(\tau) = 1$ and $\rho(\tau^-) > 0$ that ρ jumps from $\rho(\tau^-)$ to 1 at τ. Further note this implies if $F_\tau(\tau) = 0$, then ρ does not jump at τ since $\rho(\tau) = \rho(\tau^-)$.

If $F_\tau(\tau) = 0$, reputation ρ still moves. First, even if the hazard rate of default is zero (that is, $F'_\tau(\tau) = 0$), probabilist type switches cause ρ to drift towards $\epsilon/(\epsilon + \delta)$ (its unconditional long-run mean). Second, if the hazard rate of default is positive (that is, $F'_\tau(\tau) > 0$), not experiencing default increases the drift in ρ, as not defaulting is informative about the government type. Bayesian updating in this case implies that[9]

$$\rho'(\tau) = (1 - \rho(\tau))\epsilon + \rho(\tau)((1 - \rho(\tau))F'_\tau(\tau) - \delta). \tag{4}$$

Finally, consider the case where $F_0(0) = 1$. As noted above, Bayes' rule does not apply, as the probability of not observing default at exactly date 0 is zero. In this case, we let $\rho(0)$ (the belief *after* no default is observed at date zero) be a free variable. Given this belief $\rho(0)$, equations (3) and (4) continue to hold and determine the evolution of beliefs at all subsequent dates.

2.4 Prices

Let $q(\tau)$ denote the price of the bond if there has not been a default for τ periods. Given the risk neutrality assumption on the foreigners, the price solves

$$q(\tau) = \rho(\tau)q^c(\tau) + (1 - \rho(\tau))q^o(\tau), \tag{5}$$

where $q^c(\tau)$ denotes the price if there has not been a default for τ periods and the commitment type is known to be in power, and $q^o(\tau)$ denotes the price if there has not been a default for τ periods and the rational type is known to be in power. These prices must lie in $[0, 1]$ and solve the following recursion, given a default strategy F_τ for the optimizing type: First,

$$q^c(\tau) = \int_0^\infty \left(\int_0^s (i + \lambda)e^{-(i+\lambda)\tilde{s}}\mathrm{d}\tilde{s} + e^{-(i+\lambda)s}q^o(\tau + s) \right) \delta e^{-\delta s}\mathrm{d}s. \tag{6}$$

Here, the outer integral is the expectation over the first type switch. The variable s in the outer integral represents the date of the first type switch from commitment to opportunistic. The two terms in the parentheses calculate the value of the bond conditional on s. The first term is the date τ value of the coupon stream between τ and $\tau + s$. The second term is the date τ value of the remaining bond at date $\tau + s$ conditional on a type switch to an opportunistic government at

[9] For small time interval $\Delta > 0$ and a constant hazard rate of default $F'_\tau(\tau)$,

$$\rho(\tau + \Delta) \approx (1 - \delta\Delta)\frac{\rho(\tau)}{\rho(\tau) + (1 - \rho(\tau))(1 - F'_\tau(\tau)\Delta)} + \epsilon\Delta\left(1 - \frac{\rho(\tau)}{\rho(\tau) + (1 - \rho(\tau))(1 - F'_\tau(\tau)\Delta)}\right).$$

To see this, assume that the government's type stays constant on the interval $[\tau, \tau + \Delta)$ and switches at $\tau + \Delta$ from the commitment type to the opportunistic type with conditional probability $\delta\Delta$, and from the opportunistic to the commitment type with conditional probability $\epsilon\Delta$. The term $\rho(\tau)/(\rho(\tau) + (1 - \rho(\tau))(1 - F'_\tau(\tau)\Delta))$ is then the belief, conditional on no default between τ and $\tau + \Delta$, that the government is the commitment type just before $\tau + \Delta$. Thus the first term is the probability the government was the commitment type just before $\tau + \Delta$ and didn't switch at $\tau + \Delta$, and the second term is the probability the government was the opportunistic type just before $\tau + \Delta$ and did switch at $\tau + \Delta$; the two ways the government can be the commitment type at $\tau + \Delta$. The derivative of this expression with respect to Δ evaluated at $\Delta = 0$ is equation (4).

that time.

Second,

$$q^o(\tau) = \int_0^\infty \Bigg[\left(\int_0^s (i+\lambda) e^{-(i+\lambda)\tilde{s}} \mathrm{d}\tilde{s} + e^{-(i+\lambda)s} q^c(\tau + s) \right) (1 - F_\tau(\tau + s)) + \\ \int_0^s \left(\int_0^{\tilde{s}} (i+\lambda) e^{-(i+\lambda)\Delta} \mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tau + \tilde{s}) \Bigg] \epsilon e^{-\epsilon s} \mathrm{d}s. \quad (7)$$

Here, the outer integral is again the expectation over the first type switch. The terms in the square brackets calculate, again, the value of the bond conditional on s. With probability $(1 - F_\tau(\tau + s))$, default does not occur before date s, and the terms in parentheses are similar to equation (6), but this time using q^c instead of q^o. The last term handles the case where default occurs before the type switch. The outer integral of this term is the expectation over the default date $\tilde{s}$, and the inner integral calculates the value of the coupon payments up to that date.

Note that its integral form implies that $q^c(\tau)$ is continuous. Whether $q^o(\tau)$ is continuous or not depends on the strategy profile $\{F_\tau\}$.

When $F_\tau(\tau) = 0$, the above implies that $q(\tau)$ obeys the following differential equation:

$$\underbrace{\left[i + \lambda + (1 - \rho(\tau)) F_\tau'(\tau) \right]}_{\text{effective discount rate}} q(\tau) = \underbrace{(i + \lambda)}_{\text{coupon}} + \underbrace{q'(\tau)}_{\text{capital gain}} . \quad (8)$$

3 Markov Equilibria

We will focus attention on Markov equilibria where both government types follow the debt accumulation rule $H(b, q)$ (and later verify conditions such that each type wishes to do so). The definition of a Markov equilibrium is:

Definition 2. *A **Markov equilibrium** is a strategy profile for opportunistic governments $\{F_\tau\}_{\tau=0}^\infty$, together with debt, its price and reputation, (b, q, ρ), as a functions of time since last default, such that*

1. *(Foreign investors break even in equilibrium.) q and ρ satisfy (5) for some $q^c : \mathbb{R}^+ \longrightarrow [0, 1]$ and $q^o : \mathbb{R}^+ \longrightarrow [0, 1]$ that solve equations (6) and (7).*

2. *(Market beliefs are rational.) $\rho : \mathbb{R}^+ \longrightarrow [0, 1]$; equation (4) holds for all $\tau \geq 0$. For $\tau > 0$ and $\tau = 0$ if $F_0(0) < 1$, equation (3) holds if $F_\tau(\tau) > 0$.*

3. *(Debt evolution.) The amount of debt, conditional on no default, evolves according to the pre-*

specified expenditure rule:

$$b'(\tau) = H(b(\tau), q(\tau))$$

with $b(0) = 0$.

4. *(Optimizing type optimizes.) For all times since the last default* $\tau \geq 0$, *the optimizing government's continuation strategy* F_τ *maximizes its forward looking payoff taking the path of* b *and* q *as given. That is,* $\{F_\tau\}_{\tau=0}^{\infty}$ *solves the following collection of optimal control problems (indexed by* τ*):*

$$V(\tau) = \sup_{F_\tau \in \Gamma} \int_0^\infty \left(\int_\tau^t e^{-(r+\epsilon)(s-\tau)} u(C(b(s), q(s)))ds + e^{-(r+\epsilon)(t-\tau)} V(0) \right) dF_\tau(t).$$

5. *(Markov refinement.)* $\{F_\tau\}_{\tau=0}^{\infty}$ *satisfies the conditions in Definition 1.*

To clarify, Condition 4 does not allow for any "commitment" to future strategies by the optimizing type, as each opportunistic government takes as given the future paths of prices and debt. That is, altering future default probabilities does not feed into the prices that the date τ government faces. The connection between default probabilities and prices is imposed through different equilibrium conditions (in this case, Conditions 1 and 2).

4 A Markov Equilibrium

In this section, we construct a Markov equilibrium as a solution to a pair of ordinary differential equations. In the next section, we show that any Markov equilibrium must also solve these equations.

The main idea for the equilibrium construction is to conjecture that there exists a finite time since the last default, T, such that before T, an optimizing government sets a strictly positive but finite hazard rate of default (that is, $F_\tau(\tau) = 0$ and $F_\tau'(\tau) > 0$ for $\tau < T$), and consumes at a constant level $c^\star > y$. After and including time T, it defaults immediately, and thus the reputation of surviving after T is at its maximum, $\rho = 1$.

This formulation (constant consumption while default is less than certain) guarantees a constant continuation value for the optimizing type, which keeps this government type indifferent between defaulting or not. We later show such indifference is necessary for equilibrium.

Construction of $(b(\tau), q(\tau), \rho(\tau))$ **for** $\tau < T$**.** For consumption to be constant, paths of bond prices $q(\tau)$ and debt levels, $b(\tau)$, must solve

$$C(b(\tau), q(\tau)) = c^\star. \tag{9}$$

Differentiating (9) with respect to τ and using (1), we then have that $b(\tau)$ and $q(\tau)$ solve

$$b'(\tau) = H(b(\tau), q(\tau)) \tag{10}$$

$$q'(\tau) = \frac{-C_b(b(\tau), q(\tau))}{C_q(b(\tau), q(\tau))} H(b(\tau), q(\tau)). \tag{11}$$

Thus, given a value for $c^\star > y$, candidate paths of $b(\tau)$ and $q(\tau)$ are determined as the solution to the system of differential equations (10) and (11) with initial conditions $b(0) = 0$ and $q(0)$ such that $C(0, q(0)) = c^\star$. Note that since $c^\star > y$, $b'(\tau) > 0$ for all τ. Next, $q'(\tau) > 0$ as well since $C_b(b, q) = -i - \lambda(1-q) + qH_b(b, q) < 0$ since $q \leq 1$ and $H_b(b, q) \leq 0$, and $C_q(b, q) = H(b, q) + \lambda b + H_q(q, b) > 0$ since $b'(\tau) = H(b(\tau), q(\tau)) > 0$ and $H_q(q, b) > 0$ by assumption. To restate, for consumption to be constant and greater than the country's endowment, bond levels, $b(\tau)$, and bond prices, $q(\tau)$, must each be rising over time. Define T to be the date where the so-constructed bond price equals its long-run value. That is, T is such that $q(T) = \frac{i+\lambda}{i+\lambda+\delta}$ (the price associated with a constant default arrival rate of δ).

Given the candidate path of bond prices, $q(\tau)$, we can obtain the corresponding evolution of reputations, $\rho(\tau)$, that are consistent with this evolution of prices. In particular, given the path $q(\tau)$, let the path of *unconditional* arrival rates of default, $x(\tau) \equiv (1-\rho(\tau))F_\tau'(\tau)$, satisfy the pricing equation (8), now written as

$$q'(\tau) = (i + \lambda + x(\tau))q(\tau) - (i + \lambda). \tag{12}$$

Having thus obtained the path for unconditional arrival rates of default, $x(\tau)$, we can obtain the evolution of the market belief from Bayes' rule, which delivers:

$$\rho'(\tau) = (1 - \rho(\tau))\epsilon + \rho(\tau)(x(\tau) - \delta). \tag{13}$$

This is an ordinary differential equation with initial condition $\rho(0) = 0$, thus determining a candidate path $\rho(\tau)$. Thus, given a candidate $c^\star$, we have constructed candidate equilibrium objects, b, q, ρ for $\tau < T$ where T is such that $q(T) = \frac{i+\lambda}{i+\lambda+\delta}$.

Construction of $(b(\tau), q(\tau), \rho(\tau))$ **for** $\tau \geq T$**.** For $\tau \geq T$, we conjecture that the optimizing government defaults immediately and thus $\rho(\tau) = 1$ and $q^o(t) = 0$ for $\tau \geq T$. Equations (5) and

(6), together with (1), then imply

$$q(\tau) = \frac{i+\lambda}{i+\lambda+\delta} \text{ for } \tau \geq T \tag{14}$$

$$b'(\tau) = H\left(b(\tau), \frac{i+\lambda}{i+\lambda+\delta}\right) \text{ for } \tau \geq T \tag{15}$$

with the boundary condition that $b(\tau)$ be continuous at T, given the solution we have obtained above for $\tau < T$. Note by construction that $b(\tau)$ and $q(\tau)$ are continuous.

Construction of $\{F_\tau\}_{\tau=0}^{\infty}$. For $\tau < T$, the definition of Markov strategies implies that $F'_\tau(s) = F'_s(s)(1 - F_\tau(s))$ for all $s < [\tau, T)$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}s} \log(1 - F_\tau(s)) = -\frac{x(s)}{1-\rho(s)},$$

where we have used that $x(s) = (1-\rho(s))F'_s(s)$. Using a guess that $F_\tau(\tau) = 0$ for $\tau < T$, we integrate the above and obtain

$$F_\tau(s) = \begin{cases} 1 - \exp\left[-\int_\tau^s \frac{x(\hat{s})}{1-\rho(\hat{s})} d\hat{s}\right] & \text{for } s \in [\tau, T), \\ 1 & \text{for } s \geq T. \end{cases} \tag{16}$$

For $\tau \geq T$, we set $F_\tau(s) = 1$ for all $s \geq \tau$, or that the optimizing government defaults immediately if it has been weakly longer than T since the last default.

To recap, the assumption of constant consumption c^* allows us to construct a candidate path of bond levels, $b(\tau)$, and prices, $q(\tau)$, which induce this constant level of consumption by the commitment type. These bond prices then imply the unconditional default rates, $x(\tau)$, which justify them. These unconditional default rates then imply the evolution of reputation $\rho(\tau)$, and finally, the unconditional default rates and reputation determine the conditional default rates $F'_\tau(\tau)$.

A difficulty of this construction is that the constructed paths $(b(\tau), q(\tau), \rho(\tau))$, $\{F_\tau\}_{\tau=0}^{\infty}$ depend on the initial posited value of $c^\star$ (which implies the $q(0)$ that solves $c^\star = C(0, q_0)$). We next argue that the constructed bond prices $q(\tau)$ will satisfy the pricing condition in the definition of a Markov equilibrium only if $\rho(\tau)$ is continuous at T and this condition pins down $c^\star$. To see this, let $\overline{q}(\tau)$ be the bond prices consistent with our constructed default strategy $\{F_\tau\}_{\tau=0}^{\infty}$. A requirement of equilibrium will be that $q(\tau) = \overline{q}(\tau)$ for all $\tau \geq 0$. First note that $q(\tau) = \overline{q}(\tau)$ for $\tau \geq T$. This follows since for $\tau \geq T$, $F_\tau(\tau) = 1$, which implies $\rho(\tau) = 1$ and $\overline{q}^o(\tau) = 0$ for all $\tau \geq T$. Equation (6) then implies $\overline{q}^c(\tau) = \frac{i+\lambda}{i+\lambda+\delta}$ for all $\tau \geq T$. Equation (5) then implies $\overline{q}(\tau) = \rho(\tau)\overline{q}^c(\tau) = \frac{i+\lambda}{i+\lambda+\delta} = q(\tau)$, again for all $\tau \geq T$. Next, consider $\overline{q}(T^-) \equiv \lim_{\tau \to T^-} \overline{q}(\tau)$, or the price of a bond the instant before T implied by the constructed default behavior $\{F_\tau\}_{\tau=0}^{\infty}$. Here, $\overline{q}^c(T^-) = \overline{q}^c(T) = \frac{i+\lambda}{i+\lambda+\delta}$ and $\overline{q}^o(T^-) = \overline{q}^o(T) = 0$, and thus equation (5) implies $\overline{q}(T^-) = \rho(T^-)\frac{i+\lambda}{i+\lambda+\delta}$. However, by construction, $q(T^-) = \frac{i+\lambda}{i+\lambda+\delta}$. Thus, $q(T^-) = \overline{q}(T^-)$ only if $\rho(T^-) = 1$, or ρ is continuous at T.

Having established that our candidate Markov equilibrium is an equilibrium *only if* ρ is continuous at T, the following proposition shows that our candidate is an equilibrium *if* ρ is continuous at T.

Proposition 1. *For given $c^\star > y$, let $\{F_\tau\}_{\tau=0}^{\infty}$, $\{q(\tau), \rho(\tau), b(\tau)\}$ be constructed as above. If $\rho(T^-) = 1$ (or ρ is continuous at T), then $\{F_\tau\}_{\tau=0}^{\infty}$, $\{q(\tau), \rho(\tau), b(\tau)\}$ is a Markov equilibrium.*

Proof. See Appendix A. □

As Proposition 1 makes clear, to construct an equilibrium of this type, we must find $c^\star$ such that $\rho(\tau)$ is continuous at T.

For there to be multiple Markov equilibria of this type, there would have to be multiple $c^\star$ where ρ is continuous at T. We have not seen this in practice when computing examples. In practice, if $c^\star$ is chosen too low, then at the point T when $q(T) = \frac{i+\lambda}{i+\lambda+\delta}$, $\rho(T) < 1$, and if $c^\star$ is chosen too high, then at the point T when $q(T) = \frac{i+\lambda}{i+\lambda+\delta}$, $\rho(T) > 1$, with exactly one $c^\star$ such that at the point T when $q(T) = \frac{i+\lambda}{i+\lambda+\delta}$, $\rho(T) = 1$. In practice, this guarantees both the existence and uniqueness of a Markov equilibrium of this form.

Note that *nowhere* in our construction are any parameters associated with the preferences of the optimizing type. The preferences of the optimizing type — its rate of time preference and its utility function (and thus risk aversion) — are not relevant. The reason is that, in equilibrium, it faces no consumption variation either across states of nature (the realizations of arrivals of its Poisson default events) or across time. The preferences of the commitment type do not enter anywhere in the construction either. (In fact, to this point, we haven't even introduced them.) The preferences of the commitment type enter our analysis only when checking whether it will prefer to follow the borrowing rule $H(b, q)$. This, and whether the optimizing type prefers to follow H, we check for in a later section.

5 An Example

This section presents a computed example to illustrate the nature of our constructed Markov equilibrium. The parameters of our model are the endowment level y, the switching probabilities ϵ and δ, the outside world discount rate i, the coupon debt maturity parameter λ, and the net borrowing function of the commitment type $H(b, q)$ (which together imply the consumption function of the commitment type $C(b, q) = y - (i + \lambda)b + q(H(b, q) + \lambda b)$).

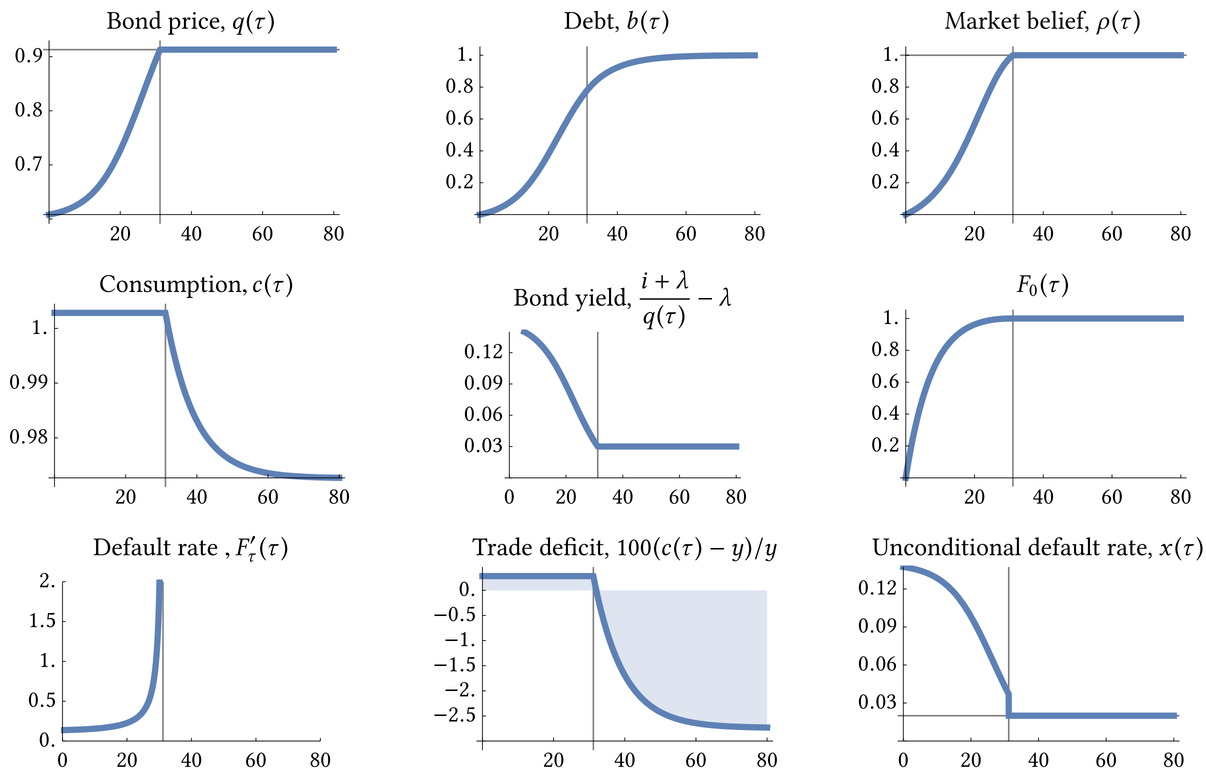

Figure 1: Equilibrium path starting from $\rho_0 = 0$ and $b_0 = 0$. H is as in equation (17). The rest of the parameters are $y = 1$, $\epsilon = 0.1$, $\delta = 0.02$, $i = 0.01$, and $\lambda = 0.2$. The value of T is represented by the vertical line.

Here, we normalize $y = 1$ and choose our other parameters relative to a unit of time being one year. Thus, if we set $\epsilon = .01$ and $\delta = .02$, this implies a 1% chance that an optimizing government dies in the next year to be replaced by a commitment government, and a 2% chance that a commitment government dies to be replaced by an optimizing government. (And thus, the country has a commitment government one-third of the time.) We set the outside world discount rate $i = .01$ and $\lambda = .2$, corresponding to a yearly principal payoff of 20% or roughly five-year debt. These imply that in the long run (after date T is reached and thus the government is certainly the commitment type), the probability of default is 2% per year (from $\delta = .02$), and thus the long-run interest rate is 3% (from $i + \delta = .03$) and the long-run bond price is $.913 = \frac{i+\lambda}{i+\lambda+\delta}$ (as opposed to a bond price of one if lending were riskless). For the commitment type's borrowing function $H(b, q)$ and its corresponding consumption function $C(b, q)$, we chose

$$H(b, q) = \left(.15 - \left(\frac{i + \lambda}{q} - \lambda\right)\right)(y - b)\,. \tag{17}$$

This is exactly what falls out of the deterministic optimization problem of a country with log utility and a discount rate of .15 who believes it can sell debt at the constant bond price q (and thus faces an interest rate of $\frac{i+\lambda}{q} - \lambda$), with the exception that we set net borrowing proportional to $(y - b)$, whereas in the deterministic problem, net borrowing is proportional to $(\frac{y}{i} - b)$.

Figure 1 displays some relevant time paths for these parameters (again, where all paths start over given a default). Here, it takes about 31 years for the market belief that the government is a commitment type to go from $\rho = 0$ to $\rho = 1$. In this time, debt goes from $b = 0$ to $b = .8$ (or a debt/GDP ratio of 80%) to, eventually, $b = 1$ (or a debt/GDP ratio of 100%), whereas the bond price goes from .6 to its long run value of .91. Consumption stays steady at about .3% above endowment for these 31 years, and then smoothly decreases over the next 30 years to about 97% of the country's endowment. The country's default rate starts at about 14% per year, decreasing to, eventually, 2% per year.

A useful consequence of analytically characterizing a Markov equilibrium is that nearly every moment of an example can be calculated as opposed to simulated. Here, for instance, once a country's interest rate reaches its long-run value $T = 31$ years after its last default, the expected time to default is $\int_0^\infty t\delta e^{-\delta t}dt = \frac{1}{\delta}$, or for these example parameters, 50 years. The average length of time to graduate after a default (call this m) is a more difficult formula, but can be expressed as

$$m = T + \int_0^T tx(t)e^{\int_t^T x(s)ds}dt,$$

which for these example parameters is a bit less than 200 years. (Recall it takes $T = 31$ years in this example go from default to graduation *conditional on not defaulting*. The probability of not defaulting for T years after a default, however, is $e^{-\int_0^T x(t)dt}$, which for these parameters is a bit less than 10%.)

6 Characterizing All Markov Equilibria

In this section, we give a tight characterization of *all* Markov equilibria such that both types follow *any* borrowing rule $H(b, q)$ satisfying Assumptions 1 through 3, and show all are of the type constructed in the previous section.

Specifically, we show in *any* Markov equilibrium, that the continuation value to the optimizing government equals a constant. (And thus, there is no value, on the equilibrium path, to having a good reputation.) This then implies that the on-path consumption of the optimizing type must also be constant. Finally, we show that if this constant on-path consumption exceeds the country's endowment (as it does for all of our computed equilibria), then there exists a date $T > 0$ such that $F_\tau(\tau) = 1$ for all $\tau \geq T$. These two characteristics (constant on-path consumption, $c^\star$, and date T such that $F_\tau(\tau) = 1$ for all $\tau \geq T$) are all we used in the previous section to construct our candidate equilibrium. Everything else about the constructed equilibrium was implied by the equilibrium conditions. Thus *all* Markov equilibria with $c^\star > y$ have the form of our constructed equilibrium from the previous section.[10]

We now turn to proving these characterizations. For a given Markov equilibrium $(\{F_\tau\}_{\tau=0}^\infty, q(\tau), \rho(\tau), b(\tau))$, let $V(\tau)$ and $c(\tau)$ denote the associated value to the optimizing government and consumption level as a function of time since the last default, τ. We first establish that $V(0) \geq \frac{u(y)}{r+\epsilon}$ and that for all $\tau \geq 0$, $V(\tau) = V(0)$.

Lemma 1. *If $(\{F_\tau\}_{\tau=0}^\infty, q(\tau), \rho(\tau), b(\tau))$ is a Markov equilibrium with associated value $V(\tau)$, $V(0) \geq \frac{u(y)}{r+\epsilon}$.*

Proof. Suppose $V(0) < \frac{u(y)}{r+\epsilon}$. From $b(0) = 0$, $c(0) = y + q(0)H(0, q(0)) \geq y$ (from $q(0) \geq 0$ and $H(0, q(0)) \geq 0$). Thus, a deviation setting $F_0(0) = 1$ ensures that consumption weakly exceeds y at all dates. □

Proposition 2. *If $(\{F_\tau\}_{\tau=0}^\infty, q(\tau), \rho(\tau), b(\tau))$ is a Markov equilibrium with associated value $V(\tau)$, for all $\tau \geq 0$, $V(\tau) = V(0)$.*

Proof. See Appendix D. □

From this proposition, it follows that the equilibrium consumption path for the optimizing type must be constant:

Lemma 2. *If $(\{F_\tau\}_{\tau=0}^\infty, q(\tau), \rho(\tau), b(\tau))$ is a Markov equilibrium with associated constant value V, then for any $\tau \geq 0$ and $\Delta > 0$ such that $F_\tau(\tau + \Delta) < 1$, $c(t) = c^\star \equiv u^{-1}\big((r+\epsilon)V\big)$ for almost all $t \in [\tau, \tau + \Delta]$.*

Proof. Consider such a τ and Δ. The fact that the optimizing type is indifferent between defaulting or not in $[\tau, \Delta]$ implies that

$$\int_\tau^{\tau+\Delta'} [u(c(t)) - (r+\epsilon)V]\, dt = 0$$

for any $\Delta' \leq \Delta$, which implies that $u(c(t)) = (r+\epsilon)V$ a.e., implying the result. □

Consider now an equilibrium where $c^\star > y$. Define $T(0) \equiv \inf\{s \geq 0 | F_0(s) = 1\}$. Note that $c^\star > y$ guarantees that $T(0) > 0$. The previous lemma guarantees that if $T(0) = \infty$, consumption of both types equals (a.e.) $c^\star > y$, which violates the bounded debt restriction. In the following proposition, we show that after $T(0)$, default always occurs immediately.

[10] In this paper, we do not rule out Markov equilibria where $c(\tau) = y$ and $b(\tau) = 0$ for all $\tau \geq 0$. However, we are unable to construct such equilibria and strongly suspect they do not exist.

Proposition 3. *Suppose* $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$ *is a Markov equilibrium with associated constant value* $V > u(y)/(r+\epsilon)$*. Then,* $F_\tau(\tau) = 1$ *for all* $\tau \geq T(0)$*.*

Proof. Suppose there exists $\tau \geq T(0)$ such that $F_\tau(\tau) < 1$. Define $\tau^\star \equiv \inf\{\tau \geq T(0) | F_\tau(\tau) < 1\}$. Hence, there exists a $\Delta > 0$ such that $c(\tau) = c^\star$ a.e. on $(\tau^\star, \tau^\star + \Delta)$. Given that $c^\star > y$, it follows that $b(\tau)$ is strictly increasing on $(\tau^\star, \tau^\star + \Delta)$. In addition, $c^\star = C(b(\tau), q(\tau))$, and thus $q(\tau)$ is strictly increasing on $(\tau^\star, \tau^\star + \Delta)$ as well.

We now established that $\rho(\tau^\star) = 1$. Consider two cases: First assume $F_{\tau^\star}(\tau^\star) = 1$. Then it is immediate, since $\rho(\tau^{\star -}) > 0$ from $T(0) > 0$. Second, assume $F_{\tau^\star}(\tau^\star) < 1$. Then, $\lim_{\tau \uparrow \tau^\star} F_\tau(\tau) = 1$ from the definition of $\tau^\star$. This implies $\rho(\tau^{\star -}) = 1$, which implies $\rho(\tau^\star) = 1$ from (3).

Given that $\rho(\tau^\star) = 1$, it follows that $q(\tau)$ is continuous at $\tau^\star$. (This is implied by (5) and the fact that q_c is always continuous.) Then, from (8),

$$\left(i + \lambda\right) q(\tau^\star) = (i + \lambda) + q'(\tau^\star),$$

which implies that $q'(\tau^\star) < 0$, contradicting $q(\tau^\star)$ strictly increasing on $(\tau^\star, \tau^\star + \Delta)$. $\square$

7 Starting Points Other than $\rho_0 = 0$ and $b_0 = 0$

To this point, we have assumed our game starts with $\rho_0 = 0$ and $b_0 = 0$. This is the relevant subgame after the first default and any subsequent defaults. We now turn to characterizing Markov equilibria for starting values of (b, ρ) other than $(0, 0)$ and use these results to establish that following borrowing rule H is indeed optimal for the optimizing type (or that the optimizing type chooses to reveal its type only by defaulting) and is optimal for the commitment type as long as it is sufficiently impatient. Finally, we show how this analysis of arbitrary starting values of debt and reputation allows for the consideration of probability zero shocks.

Consider Figure 2. In the $(0, 0)$ game, the state variables (b, ρ) start at $(0, 0)$ and over time (if no default) move to the northeast along the thick blue line (the $(0, 0)$ equilibrium manifold) with both debt and reputation increasing until debt reaches its date T level, $b(T)$, in which case reputation is at its maximum (one), but debt continues to increase until reaching its steady state. (Every time there is a default, the state variables return to $(0, 0)$.)

Next, partition (b, ρ) space into starting values (b_0, ρ_0) such that the government's initial reputation ρ_0 is *equal to* (on the blue line), *greater than* (above the blue line), and *less than* (below the blue line) what its reputation is for that level of debt in the $(0, 0)$ subgame. More formally, note first that since $b(\tau)$ in the $(0, 0)$ subgame is strictly increasing over time, there is a one-to-one mapping between debt levels b and times since the last default τ, and thus we can define $\tau^*(b)$ to be the amount of time it takes to reach debt b in the $(0, 0)$ subgame. Since any Markov

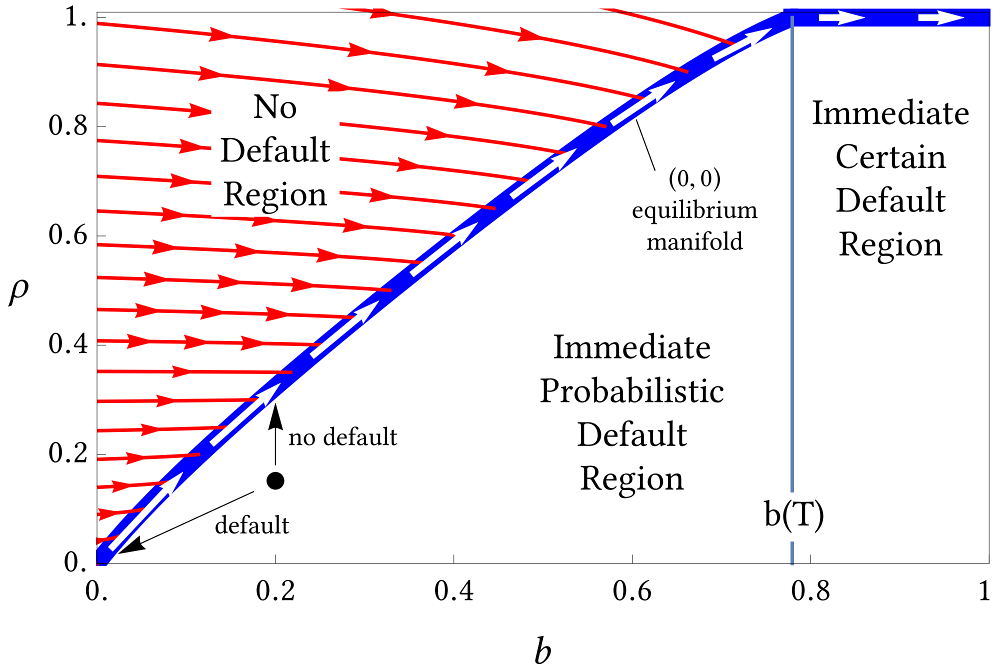

Figure 2: *Default regions. The thick blue line depicts the equilibrium manifold starting from* $b_0 = 0$ *and* $\rho_0 = 0$*. At any starting point above the manifold, no default occurs, and* (b, ρ) *moves along a red line until reaching the manifold. At any starting point below the blue line, the equilibrium jumps to the blue line if no immediate default occurs. To the right of* $b(T)$*, there is immediate certain default by the optimizing type, independently of* ρ*. To the left of* $b(T)$*, immediate default occurs probabilistically.*

equilibrium defines functions $b(\tau)$ and $\rho(\tau)$, the equilibrium manifold is then represented by the function $\rho(\tau^*(b))$.

Next, consider (b_0, ρ_0) such that $\rho_0 = \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is exactly what it is in the $(0, 0)$ subgame when it has debt b_0 (or (b_0, ρ_0) is on the blue line). In these cases, assume the optimizing type follows its strategy from the $(0, 0)$ subgame, but starting as if it has been $\tau^*(b_0)$ periods since the last default. Since following this strategy is optimal in the $(0, 0)$ subgame, it is optimal starting from (b_0, ρ_0).

Next, assume $\rho_0 > \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is strictly greater than what it is in the $(0, 0)$ subgame when it has debt b_0 (or (b_0, ρ_0) is above the blue line). Here, we propose the optimizing type sets $F_0(t) = 0$ for a specific amount of time t^* (which depends on (b_0, ρ_0)). Its reputation ρ at date $t < t^*(b_0, \rho_0)$ is then

$$\hat{\rho}(t) = \frac{\epsilon}{\epsilon + \delta} + e^{-(\epsilon+\delta)t}\left(\rho_0 - \frac{\epsilon}{\epsilon + \delta}\right),$$

which converges continuously to $\frac{\epsilon}{\epsilon+\delta}$. Since in the $(0, 0)$ subgame, $\rho(\tau)$ moves continuously from zero to one, this ensures there exists (t^*, τ) such that $\hat{\rho}(t^*) = \rho(\tau)$. From date t^* on then, we propose the optimizing type follows the $(0,0)$ equilibrium starting as if it had been τ periods since the last default. Graphically, the state variables (b, ρ) move continuously to the east along the red line associated with (b_0, ρ_0) until hitting the blue line, where from there the game follows the $(0,0)$ equilibrium as if it has been τ periods since a default. Since from time zero to time t, the bond price q is strictly greater than the bond price for that level of debt in the $(0,0)$ equilibrium (since the probability of default is zero for some time), consumption along this path is strictly greater than the consumption of the optimizing type in the $(0,0)$ subgame, ensuring the optimizing type is willing to set $F_0(t) = 0$ for $t < t^*$.

Finally, suppose $\rho_0 < \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is strictly less than what it is in the $(0,0)$ subgame when it has debt b_0. Here, if $\rho_0 > 0$ and $b_0 < b(T)$ (labeled the "Immediate Probabilistic Default Region" in Figure 2), we propose the optimizing type immediately defaults with probability γ such that

$$\rho(\tau^*(b_0)) = \frac{\rho_0}{\rho_0 + (1-\rho_0)(1-\gamma)}.$$

Since $0 < \rho_0 < \rho(\tau^*(b_0))$, then $\gamma \in (0,1]$, and this default behavior ensures that (b, ρ) jumps either to $(0,0)$ (in the case of immediate default) or $(b_0, \rho(\tau^*(b_0)))$ (in the case of no immediate default). The optimizing type then follows the strategy from the $(0,0)$ game starting from either $\tau = 0$ or $\tau = \tau^*(b_0)$ depending on whether it immediately defaulted. It is willing to set γ between zero and one because its continuation value is the same in either case.[11] If $\rho_0 > 0$ and $b_0 \geq b(T)$ (labeled the "Immediate Certain Default Region" in Figure 2), we propose the opportunistic government immediately defaults with probability one. Under this strategy, reputation ρ immediately jumps to one if no default. Here, the opportunistic government strictly prefers to default.

If $\rho_0 = 0$, we propose the optimizing type immediately defaults with probability one. Here, Bayes' rule doesn't apply for calculating beliefs conditional on not defaulting; thus we are free to set the belief. If $b_0 \leq b(T)$, setting ρ conditional on no default to $\rho(\tau^*(b_0))$ again ensures the optimizing type's payoff is the same regardless of whether he defaults or not and is thus willing to set γ to one.[12] If $b_0 > b(T)$, the optimizing type finds it strictly optimal to default.

[11]Note that if the strategy calls for γ high enough that ρ jumps above $\rho(\tau^*(b_0))$ if no default, the optimizing government will find it optimal to deviate and set $\gamma = 0$. If the strategy calls for γ low enough so that ρ fails to reach $\rho(\tau^*(b_0))$ if no default, the strategy calls for another immediate probabilistic default to reach $\rho(\tau^*(b_0))$, which is the same thing as choosing γ such that $\rho(\tau^*(b_0))$ is reached immediately if no default.

[12] Again note that if we set ρ after no default greater than $\rho(\tau^*(b_0))$, the optimizing type will find it optimal to deviate and not default, and if we set ρ after no default less than $\rho(\tau^*(b_0))$, the strategy again calls for another immediate (probabilistic) default.

7.1 Optimality of following $H(b, q)$

This characterization of play for arbitrary (b, ρ) starting points allows us to examine whether both types should follow the arbitrary rule $H(b, q)$ determined by the expectations of outside lenders. Again, we assume the borrowing country is supposed to follow H whenever $\rho \geq \rho(\tau^*(b))$ (which occurs with equality in the $(0, 0)$ subgame and as a strict inequality for sufficiently high ρ_0 relative to b_0) and that outside lenders believe $\rho = 0$ whenever a country attempts to borrow differently.

First consider the optimizing type. In any Markov equilibrium, its value is the same for all (b_0, ρ_0) such that $\rho_0 \leq \rho(\tau^*(b_0))$ and strictly above this common value when $\rho_0 > \rho(\tau^*(b_0))$. If it deviates when $\rho \geq \rho(\tau^*(b))$ and borrows differently from $H(b, q)$, its reputation ρ becomes zero, and its value is either unchanged (in the case where $\rho_0 = \rho(\tau^*(b_0))$) or declines (in the case where $\rho_0 > \rho(\tau^*(b_0))$). Thus, the optimizing type will always find it optimal to follow the borrowing rule H, choosing to reveal its type only by defaulting.

Next, consider the commitment type. To this point, we have not specified its preferences. Since we have assumed that if a country attempts to borrow differently than H, it is assumed to be certainly the optimizing type, and the optimizing is supposed to default immediately with probability one whenever $\rho = 0$ and $b > 0$, this implies a bond price of zero immediately after a country fails to follow H (which is the same as not being able to borrow). Thus, consider a commitment type that deviates from H for an interval of time $[t, t + \Delta]$. During this period, ρ and q remain at 0, and the commitment type must simply consume its endowment while making its coupon payments, reducing its debt at the rate λ.

Suppose then at any point in the game where it has debt $b > 0$, a commitment type with bounded utility function u^c and discount factor r^c considers using this strategy to pay off Δ of its debt, taking $T(\Delta) = \frac{-\ln(1-\frac{\Delta}{b(t)})}{\lambda}$ units of time to do so. Its value from following this strategy (as a function of its current debt, b, and Δ) is

$$\hat{V}^c(b, \Delta) \equiv \int_0^{T(\Delta)} e^{-r^c s} u^c(y - (i + \lambda)e^{-\lambda s} b) ds + e^{-r^c T(\Delta)} V^c(b - \Delta),$$

where $V^c(b) = \int_{\tau^*(b)}^{\infty} e^{-r^c(s-\tau^*(b))} u(c(s)) ds$, or $V^c(b)$ is the value to the commitment type of following the equilibrium as if it has been $\tau^*(b)$ periods since the last default. Here, the first integral is the commitment type's payoff between the date it starts to deviate and the date it stops, and the second integral is its payoff from then on. At this date, since it starts following the rule H and doesn't default (since it can't), its reputation jumps to $\rho(\tau^*(b - \Delta))$, and it is back on the equilibrium path as if it has been $\tau^*(b - \Delta)$ periods since the last default. A necessary and sufficient condition for the commitment type being willing to set $\Delta = 0$ for all b is that $\frac{\partial \hat{V}^c}{\partial \Delta}|_{\Delta=0} \leq 0$ for all

b. Here,

$$\frac{dV^c(b)}{db} = \tau^{*'}(b)[r^c V^c(b) - u(c(\tau^*(b))]$$

and

$$\frac{\partial \hat{V}^c(b,\Delta)}{\partial \Delta}|_{\Delta=0} = \frac{1}{\lambda b}\Big[[u(y-(i+\lambda)b) - u(c(\tau^*(b)))] - \frac{dV^c(b)}{db}[b'(\tau^*(b)) + \lambda b]\Big].$$

That $\lim_{r^c\to\infty} r^c V^c(b) = u(c(\tau^*(b))$ implies $\lim_{r^c\to\infty} \frac{dV^c(b)}{db} = 0$. Since $[u(y-(i+\lambda)b)-u(c(\tau^*(b)))] < 0$ and $b'(\tau)$ and b are uniformly bounded, for any given borrowing rule $H(b,q)$, there exists r^c sufficiently high such that the commitment type is willing to follow it.

7.2 Probability zero shocks and asymmetric responses

Our analysis of arbitrary starting values of debt and reputation allows for the consideration of probability zero shocks. We show here that the response of the economy is asymmetric. Positive surprises generate subsequent periods of no default. Negative surprises generate an immediate (possibly probabilistic) default.

The logic is simple: Suppose a surprise permanent shock hits the level of the endowment flow y, moving it from y to $\hat{y}$. Such a shock changes neither the country's level of debt b nor its reputation ρ. But the shock does generally change the $(0,0)$ equilibrium manifold. In particular, a good shock to y moves the manifold down — having a higher endowment flow implies a lower necessary (or equilibrium) reputation for each level of debt. Thus, the response to a good surprise endowment shock is exactly the same as starting the game with (b_0, ρ_0) above the equilibrium manifold, implying a positive length of time of no default (with this length of time higher the bigger the shift in the equilibrium manifold).

Alternatively, if the country receives a bad endowment shock, again (b,ρ) doesn't change, but the $(0,0)$ manifold moves up — having a lower endowment flow implies a higher necessary (or equilibrium) reputation for each level of debt. Thus, the response to a bad surprise endowment shock is exactly the same as starting the game with (b_0, ρ_0) *below* the equilibrium manifold, implying an immediate probabilistic default to either reset (b,ρ) to $(0,0)$ (in the case of default) or raise ρ to the new level appropriate to b (in the case of no default), with a higher probability of default the more the manifold shifts.

8 Closed-Form Solution

In this section, we present a class of assumptions where our Markov equilibrium can be solved in closed form. In particular, assume $\lambda = 0$, or bonds are console bonds, and $H(b,q) = \left(r^* - \frac{i}{q}\right)\left(\frac{y}{i} - b\right)$. Here, $H(b,q)$ is exactly how a country with log preferences which discounts the future at rate r^*

borrows if it faces a constant bond price q now and into the future. We assume that the implied discount factor is sufficiently high, $r^\star > i+\delta+\epsilon$, so that there is borrowing in equilibrium. Then, we have the following result:

Lemma 3 (A closed-form solution)**.** *Suppose $\lambda = 0$ and $H(b,q) = \left(r^* - \frac{i}{q}\right)\left(\frac{y}{i} - b\right)$. Then,*

$$T = \frac{\log\left(\frac{r^*-i-\delta}{\epsilon}\right)}{r^* - i - \delta - \epsilon}, \qquad c^\star = y\left(1 + \frac{r^* - i - \delta}{i+\delta} e^{-r^* T}\right)$$

and

$$\begin{aligned} q(\tau) &= \frac{i}{r^*}\left(1 + \frac{r^* - i - \delta}{i + \delta} e^{-r^*(T-\tau)}\right) \\ \rho(\tau) &= \frac{\epsilon}{r^* - i - \delta - \epsilon}\left(e^{(r^*-i-\delta-\epsilon)\tau} - 1\right) \\ F_\tau'(\tau) &= \frac{(r^* - i)(r - i - \delta - \epsilon)}{r^* - i - \delta - \epsilon e^{(r^*-i-\delta-\epsilon)\tau}} \end{aligned}$$

for all $\tau < T$; and

$$b(\tau) = \begin{cases} \frac{(e^{-r^*T} - e^{-r^*(T-\tau)})(r^*-i-\delta)}{(1-e^{-r^*(T-\tau)})(r^*-i-\delta)-r^*}\frac{y}{i} & \text{for } \tau \le T, \\ (1 - e^{-(r^*-i-\delta)(\tau-T)})\frac{y}{i} + e^{-(r^*-i-\delta)(\tau-T)} b(T) & \text{for } \tau > T. \end{cases}$$

Proof. See Appendix E. □

Having a closed-form solution allows for easy comparative statics. For instance, it allows us to examine the role that ϵ, the switching propensity from an optimizing type to a commitment type, plays. From our closed-form solution for T, it is apparent that T increases as ϵ decreases, with T going to infinity as ϵ goes to zero; that is,

$$\frac{dT}{d\epsilon} < 0, \text{ and } \qquad \lim_{\epsilon \to 0} T = \infty.$$

Thus, as ϵ falls, the equilibrium amount of time before certain default, T, increases. Note further that $c^\star$ goes to y as T goes to infinity (or as ϵ goes to zero). Hence, the trade deficits an optimizing type can achieve ($c^\star - y$) are reduced as ϵ decreases, going to zero in the limit as ϵ approaches 0. Without a non-negligible probability of being replaced by a commitment type, given our Markov assumption on strategies, it is impossible for an opportunistic type to support a non-negligible trade deficit.

This finding makes intuitive sense. The rent that an optimizing government can extract from future commitment governments depends on the arrival rate of such future governments. If such a future commitment government can be expected to arrive only very far out in the future, the rent that an optimizing government can extract from such a prospect vanishes. (In addition, note that $q(0) = \frac{i}{r^\star} c^\star$ and thus bond prices inherit the properties of $c^\star$. In this case, a lower ϵ reduces bond prices.)

These closed-form solutions also allow the consideration of changing r^*, the discount rate of the commitment type. In particular, if the commitment type becomes more impatient, the equilibrium amount of time before certain default, T, decreases, and thus $c^\star$ and rent extraction increase. Here again, the impatience of possible future commitment types — their willingness to borrow at rates that reflect the possibility of a defaulting opportunistic type — is what allows the opportunistic type to extract rents. Hence the greater this impatience, the greater these extracted rents.

9 Conclusion

In this paper, we presented a tractable sovereign debt model where the borrower's reputation and its interaction with default events generate dynamics of debt and asset prices that are consistent with several facts.

In our model, a government that defaults loses its reputation, and it takes periods of borrowing and not defaulting to eventually restore it. During these periods, bond prices are low and default frequencies are high, as in the data. Further, relative to countries that have not recently defaulted, debt levels are low. In fact, in our model, as in the data, countries with low debt levels face relatively high interest rates, a phenomenon referred to as "debt intolerance." In our model, a country can "graduate" into the set of "debt-tolerant" countries by not defaulting for a sufficiently long period of time, as perhaps Mexico has done by not defaulting since the 1980s.

In the data, default is less than fully predictable and somewhat untied to fundamentals. Recent work has emphasized this fact as an argument for introducing features that lead to multiple equilibria in the standard sovereign debt model. In our environment, such an outcome arises naturally. Equilibrium default in our model is necessarily random, both in our baseline model and in our consideration of when a country is hit by a bad shock. Such randomness is a fundamental ingredient for the dynamics of learning and reputation.

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A Proof of Proposition 1

Proof. We proceed to argue that the candidate functions $\{F_\tau\}_{t=0}^{\infty}$, $\{q(\tau), \rho(\tau), b(\tau)\}$ such constructed satisfy the five conditions for equilibrium stated in Definition 2.

- Condition 3 holds by construction, given that (10) and (15) hold.

- For condition 4 (optimizing type optimizes), first we show that $c(\tau)$ is continuous at all τ and weakly decreasing for $\tau \geq T$. To start, note that

$$C(b, q) = y - (i + (1 - q)\lambda)b + qH(b, q). \tag{18}$$

  Using that $q(\tau) \leq 1$, the above implies that $b'(\tau) = H(b(\tau), q(\tau)) > 0$ for all $\tau < T$, since $c^\star = C(b(\tau), q(\tau)) > y$. It follows that $b'(T) = H\left(b(T), \frac{i+\lambda}{i+\lambda+\delta}\right) > 0$ given that $b(\tau)$ and $q(\tau)$ are continuous at T and that H is continuous. Thus, $b'(\tau)$ starts positive at $\tau = T$ and moves continuously over time since $q(\tau)$ is constant. Thus, $b'(\tau)$ cannot become negative; if $b'(\tau) = 0$ for some $\tau \geq T$, then since q is constant, b remains constant. Thus, $b(\tau)$ is weakly increasing for $\tau \geq T$. Since $b(\tau)$ is weakly increasing for $\tau \geq T$, $c(\tau) \equiv C(b(\tau), q(\tau))$ is weakly decreasing for $\tau \geq T$ as $q(\tau)$ is constant and less than one. Note that $c(\tau)$ is continuous at T, as both $b(\tau)$ and $q(\tau)$ are.

  That $c(\tau)$ is continuous for all τ, constant for all $\tau < T$ (by construction), and weakly decreasing for $\tau > T$ implies that the conjectured default strategy for the optimizing type is optimal. That is, the optimizing type consumes $c^\star$ at all times, and as a result it is indifferent between defaulting or not before $\tau \leq T$ and (weakly) prefers to default for $\tau > T$, thus ensuring that Condition 4 is satisfied.

- We now turn to show Condition 2 (market beliefs, $\rho(\tau)$, are rational).

  When $F_\tau(\tau) > 0$, rationality of beliefs requires that equation (3) holds (if $\rho(\tau) > 0$). For $\tau \geq T$, $\rho(\tau) = 1$ and $F_\tau(\tau) = 1$, and immediately equation (3) is satisfied for $\tau > T$. For $\tau = T$, equation (3) also holds if $\rho(T^-) > 0$ and does not apply if $\rho(T^-) = 0$.

  For $\tau < T$, by construction $\rho'(\tau) = (1 - \rho(\tau))\epsilon + \rho(\tau)(x(\tau) - \delta)$, ensuring Bayes' rule holds locally given the conjectured default behavior $F'_\tau(\tau) = x(\tau)/(1 - \rho(\tau))$. What remains to be shown is that $\rho(\tau) \in [0, 1]$ for all $\tau < T$.

  To show this, note $b'(\tau) = H(b(\tau), q(\tau)) > 0$, as argued above. Note that $C_b(b, q) = -(i + (1 - q)\lambda)b + qH_b(b, q) \leq 0$. Given that $C_q \geq 0$ by Assumption 1, it follows from equation (11) that $q'(\tau) \geq 0$.

Next we argue that

$$q(\tau) < q(T) = \frac{i+\lambda}{i+\lambda+\delta} \text{ for } \tau < T. \tag{19}$$

To show this, suppose not. Given that $q(\tau)$ is weakly increasing for $\tau < T$, this implies there exists $\tau_0 < T$ and $\tau_0 < \tau_1 < T$ such that $q(\tau_0) = q(\tau_1) = q(T)$. But then, $c^\star = C(b_0, \bar{q}) = C(b_1, \bar{q})$ for $b_0 = b(\tau_0) < b_1 = b(\tau_1)$ and $\bar{q} = q(T)$. Now

$$C(b_0, \bar{q}) - C(b_1, \bar{q}) = (i + (1-\bar{q})\lambda)(b_1 - b_0) + \bar{q}\big(H(b_0, \bar{q}) - H(b_1, \bar{q})\big) > 0,$$

a contradiction.

From (18), we have that $C(0, q(0)) = y + q(0)H(0, q(0)) = c^\star > y$, and thus $q(0) > 0$. Given that we showed above that $q(\tau)$ is increasing for all $\tau < T$, it follows that $q(\tau) > 0$ for $\tau < T$. Using equation (12), we can solve for $x(\tau)$ and obtain that

$$x(\tau) = \frac{(i+\lambda)(1-q(\tau)) + q'(\tau)}{q(\tau)} \geq 0 \text{ for all } \tau < T. \tag{20}$$

Note that given that $q(\tau) < q(T) = \frac{i+\lambda}{i+\lambda+\delta}$, by (19) and that $q'(\tau) \geq 0$ for $\tau < T$, it follows that

$$x(\tau) \geq (i+\lambda)\left(\frac{1}{q(\tau)} - 1\right) > (i+\lambda)\left(\frac{i+\lambda+\delta}{i+\lambda} - 1\right) = \delta. \tag{21}$$

From (13), we have that

$$\begin{aligned} \rho'(\tau) &= (1-\rho(\tau))\epsilon + \rho(\tau)(x(\tau) - \delta) \\ &\geq (1-\rho(\tau))\epsilon. \end{aligned}$$

As a result, $\rho(\tau)$ is increasing for $\tau < T$ and strictly increasing as long as $\rho(\tau) < 1$. That $\rho(\tau)$ is assumed continuous at T then ensures $\rho(\tau) \in [0,1]$ for all τ.

- Condition 1. (Foreign investors break even in equilibrium.) In our construction, we derive the *unconditional* bond price $q(\tau)$, but our definition of a Markov equilibrium is in terms of *conditional* prices $q^c(\tau)$ and $q^o(\tau)$.

  The existence of q^o and q^c, given $\{F_\tau\}_{\tau=0}^\infty$, follows from a contraction mapping developed in Appendix B. We next need to show that such q^o and q^c satisfy (5).

Let $\hat{q}$ be defined as

$$\hat{q}(\tau) \equiv \rho(\tau)q^c(\tau) + (1 - \rho(\tau))q^o(\tau).$$

Continuity of q^o and q^c (shown in Appendix C), together with continuity of ρ, guarantees that $\hat{q}$ is continuous.

Note that $\hat{q}(\tau) = q^c(\tau) = \frac{i+\lambda}{i+\lambda+\delta} = q(\tau)$ for $\tau \geq T$, where the second equality follows from the proposed F_τ.

Taking derivatives of the integral forms (6) and (7), together with (4) and the definition of $x(\tau)$, implies that

$$\hat{q}'(\tau) = (i + \lambda + x(\tau))\hat{q}(\tau) - (i + \lambda) \tag{22}$$

for $\tau < T$.

Note that our construction implies that $x(\tau)$ is continuous on $[0, \tau)$. In addition, $x(T^-)$ is finite. The latter follows from (20), $q(\tau) \geq q(0) > 0$, and $q'(T^-)$ finite (which follows equation (11) evaluated at $\tau = T$).

Hence, $\hat{q}$ solves an initial value problem (IVP): it satisfies equation (22) with boundary condition $\hat{q}(T) = q(T)$. This IVP is a first-order linear ordinary differential equation on $[0, T]$ with time dependent but continuous coefficients. Hence, it has a unique solution in $[0, T]$.

Given that q solves the same IVP, it follows that q and $\hat{q}$ are the same.

- Condition 5 (Each F_τ is a cumulative distribution function and theay are consistent with each other.) First we need to show that each $F_\tau \in \Gamma$. For $\tau < T$, the function defined in equation (16) lies in $[0, 1]$ and is differentiable and increasing, given that $x(s) \geq 0$ and $\rho(s) < 1$ for $s < T$ (from the arguments in the proof of Condition 2 above). Given that $F_\tau(s) = 1$ for $s > T$, it follows that $F_\tau \in \Gamma$. For $\tau \geq T$, $F_\tau \in \Gamma$ by construction.

  Next we argue that condition 2 holds. To see this, note that the condition holds for any $s \geq T$, as $F_\tau(s) = F_m(s) = 1$ for all τ, m. And for $s < T$, the condition holds given the exponential form in (16).

This completes the proof. □

B Existence of q^o and q^c Given $\{F_\tau\}_\tau^\infty$

Here we provide the argument that for a given $\{F_\tau\}_{\tau=0}^\infty$, there exists a unique q^o and q^c that satisfy the integral equations described in the main body of the paper. This shows that, given an $\{F_\tau\}$, we can indeed always construct the q^o and q^c.

Instead of working with vector-valued operators, the idea of the proof is to substitute the equation for q^c into q^o. In this way, we obtain that q^o is a fixed point of the following operator T:

$$T\left[f\right](\tau) = \int_{\tau}^{\infty} \left\{ \left(H^0(s,\tau) + e^{-(i+\lambda)(s-\tau)} \int_{s}^{\infty} \left(H^0(\tilde{s},s) + e^{-(i+\lambda)(\tilde{s}-s)} f(\tilde{s}) \right) \delta e^{-\delta(\tilde{s}-s)} \mathrm{d}\tilde{s} \right) \times \right.$$
$$\left. (1-F_\tau(s)) + \int_{\tau}^{s} H^0(\tilde{s},\tau) \mathrm{d}F_\tau(\tilde{s}) \right\} \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s, \quad (23)$$

where recall that $H^0(s,\tau) \equiv 1 - e^{-(i+\lambda)(s-\tau)}$.

Manipulating this expression, we obtain

$$T\left[f\right](\tau) = \int_{\tau}^{\infty} \left\{ \left(\int_{s}^{\infty} \left(H^0(\tilde{s},\tau) + e^{-(i+\lambda)(\tilde{s}-\tau)} f(\tilde{s}) \right) \delta e^{-\delta(\tilde{s}-s)} \mathrm{d}\tilde{s} \right) (1-F_\tau(s)) + \right.$$
$$\left. \int_{\tau}^{s} H^0(\tilde{s},\tau) \mathrm{d}F_\tau(\tilde{s}) \right\} \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s. \quad (24)$$

Let B be the space of bounded functions $f : \mathbb{R}^+ \longrightarrow [0,1]$ with the sup norm. We make the following observations about T, for a given $\{F_\tau\}$:

1. T maps B into itself.

   The argument is as follows. Consider the constant function, c. Then

$$\int_{s}^{\infty} \left(H^0(\tilde{s},\tau) + e^{-(i+\lambda)(\tilde{s}-\tau)} c \right) \delta e^{-\delta(\tilde{s}-s)} \mathrm{d}\tilde{s} = 1 + \frac{\delta e^{-(i+\lambda)(s-\tau)}}{i+\lambda+\delta}(c-1).$$

   Hence,

$$\begin{aligned} T[f](\tau) \le T[1](\tau) &= \int_{\tau}^{\infty} \left\{ (1-F_\tau(s)) + \int_{\tau}^{s} H^0(\tilde{s},\tau) \mathrm{d}F_\tau(\tilde{s}) \right\} \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s \\ &\le \int_{\tau}^{\infty} \left\{ (1-F_\tau(s)) + \int_{\tau}^{s} \mathrm{d}F_\tau(\tilde{s}) \right\} \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s \le 1, \end{aligned}$$

   where the second inequality follows from $H^0 \le 1$.

Also,

$$T[f](\tau) \geq T[0](\tau) = \int_\tau^\infty \left\{ \left(1 - \frac{\delta e^{-(i+\lambda)(s-\tau)}}{i+\lambda+\delta}\right)(1 - F_\tau(s)) + \int_\tau^s H^0(\tilde{s}, \tau)\mathrm{d}F_\tau(\tilde{s}) \right\} \epsilon e^{-\epsilon(s-\tau)}\mathrm{d}s.$$
$$\geq 0$$

where the last inequality follows from

$$1 - \frac{\delta e^{-(i+\lambda)(s-\tau)}}{i+\lambda+\delta} \geq 1 - \frac{\delta}{i+\lambda+\delta} \geq 0$$

and $H^0 \geq 0$.

2. T is a monotone operator. This is straightforward.

3. T satisfies discounting. Consider a $c > 0$. Then,

$$T[f+c](\tau) = T[f](\tau) + \underbrace{\int_\tau^\infty (1 - F_\tau(s))e^{-(i+\lambda)(s-\tau)}\epsilon e^{-\epsilon(s-\tau)}\mathrm{d}s}_{\in\left[0, \frac{\epsilon}{i+\lambda+\epsilon}\right]} \times \frac{\delta}{i+\delta+\lambda} \times c \tag{25}$$

$$\leq T[f](\tau) + \beta c \tag{26}$$

where $\beta = \frac{\delta}{i+\delta+\lambda} < 1$. And thus, T satisfies discounting.

Points (1)-(3) imply that T satisfies Blackwell sufficient conditions, and thus it is a contraction mapping. As a result, there exists a unique solution to $T[q^o] = q^o$. Given a q^o, we can use the same argument to argue that there exists a unique q^c associated with it.

C Continuity of the Price

Consider first $q^c(\tau)$. We can show that $q^c(\tau)$ is continuous for the entire domain independent of $\{F_\tau\}$. From the paper,

$$q^c(\tau) = \int_0^\infty \left(\int_0^s (i+\lambda)e^{-(i+\lambda)\tilde{s}}\mathrm{d}\tilde{s} + e^{-(i+\lambda)s}q^o(\tau+s) \right) \delta e^{-\delta s}\mathrm{d}s. \tag{27}$$

Making a change of indexes, we get

$$q^c(\tau) = \int_\tau^\infty \left(\int_\tau^s (i+\lambda) e^{-(i+\lambda)(\tilde{s}-\tau)} \mathrm{d}\tilde{s} + e^{-(i+\lambda)(s-\tau)} q^o(s) \right) \delta e^{-\delta(s-\tau)} \mathrm{d}s \tag{28}$$

$$= e^{(i+\lambda+\delta)\tau} \int_\tau^\infty \left(\int_\tau^s (i+\lambda) e^{-(i+\lambda)\tilde{s}} \mathrm{d}\tilde{s} + e^{-(i+\lambda)s} q^o(s) \right) \delta e^{-\delta s} \mathrm{d}s. \tag{29}$$

This latter function is continuous in τ, as it is the product of two continuous functions of τ.

Consider now $q^o(\tau)$. In this case, the continuity of $q^o(\tau)$ cannot be guaranteed independently of $\{F_\tau\}$.

However, consider the family $\{F_\tau\}$ that satisfies (16). In this case,

$$q^o(\tau) = \int_0^\infty \left[\left(\int_0^s (i+\lambda) e^{-(i+\lambda)\tilde{s}} \mathrm{d}\tilde{s} + e^{-(i+\lambda)s} q^c(\tau+s) \right) (1 - F_\tau(\tau+s)) + \int_0^s \left(\int_0^{\tilde{s}} (i+\lambda) e^{-(i+\lambda)\Delta} \mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tau+\tilde{s}) \right] \epsilon e^{-\epsilon s} \mathrm{d}s. \tag{30}$$

Doing the change in indexes we did above,

$$q^o(\tau) = \int_\tau^\infty \left[\left(\int_\tau^s (i+\lambda) e^{-(i+\lambda)(\tilde{s}-\tau)} \mathrm{d}\tilde{s} + e^{-(i+\lambda)(s-\tau)} q^c(s) \right) (1 - F_\tau(s)) + \int_\tau^s \left(\int_\tau^{\tilde{s}} (i+\lambda) e^{-(i+\lambda)(\Delta-\tau)} \mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tilde{s}) \right] \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s. \tag{31}$$

And simplifying a little bit:

$$q^o(\tau) = \int_\tau^\infty \left[H^0(s,\tau)(1 - F_\tau(s)) + \int_\tau^s H^0(\tilde{s},\tau) \mathrm{d}F_\tau(\tilde{s}) \right] \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s \tag{32}$$

where $H^0(s,\tau) \equiv \int_\tau^s (i+\lambda) e^{-(i+\lambda)(\tilde{s}-\tau)} \mathrm{d}\tilde{s} = 1 - e^{-(i+\lambda)(s-\tau)}$ is a positive, bounded, and continuous function of τ and s.

Given our conjectured $\{F_\tau\}$, we have that $q^o(\tau) = 0$ for $\tau \geq T$. In addition, for any $\tau < T$, we can write the above as

$$q^o(\tau) = \times \int_\tau^T \left[H^0(s,\tau)(1 - F_\tau(s)) + \int_\tau^s H^0(\tilde{s},\tau) \mathrm{d}F_\tau(\tilde{s}) \right] \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s.$$

Using that F_τ is a cumulative distribution function, and thus integrates to one, we have

$$|q^o(\tau)| \leq \int_\tau^T |H^0| \epsilon e^{-\epsilon(s-\tau)} \mathrm{d}s.$$

where $|H^0|$ is the maximum absolute value of H^0.

And as τ approaches T, we have that $\lim_{\tau \to T-} q^o(\tau) \to 0$. And thus $q^o(\tau)$ is continuous at T.

D Proof of Proposition 2

Proof. The proof proceeds in four steps. We first prove three preliminary results and then the main result:

1. $V(\tau)$ is continuous.
2. $V(\tau) \geq V(0)$ for all $\tau \geq 0$.
3. For all $\tau \geq 0$, there exists $t \geq \tau$ such that $V(t) = V(0)$,
4. For all $\tau \geq 0$. $V(\tau) = V(0)$.

1) $V(\tau)$ is continuous.

First note that $V(\tau)$ satisfies

$$\begin{aligned} V(\tau) &= \sup_{T \geq 0} \int_\tau^{\tau+T} e^{-(\rho+\epsilon)(s-t)} u(c(s)) ds + e^{-(\rho+\epsilon)T} V(0) \\ &= \int_\tau^{\tau+T(\tau)} e^{-(\rho+\epsilon)(s-\tau)} u(c(s)) ds + e^{-(\rho+\epsilon)T(\tau)} V(0) \end{aligned}$$

for some $T(\tau) \in \mathbb{R}_+ \cup \{+\infty\}$.[13] We then have that, for $\Delta > 0$,

$$\begin{aligned} V(\tau+\Delta) - V(\tau) &\geq \int_{\tau+\Delta}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} u(c(s)) ds + e^{-(\rho+\epsilon)T(\tau)} V(0) - V(\tau) \\ &= \int_{\tau+\Delta}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} u(c(s)) ds - \int_\tau^{\tau+T(\tau)} e^{-(\rho+\epsilon)(s-\tau)} u(c(s)) ds, \end{aligned}$$

[13]In particular, $T(\tau) = \inf\{t \geq \tau | F_t'(t) > 0\}$.

where the inequality uses the weak suboptimality of $T(\tau)$ at $\tau+\Delta$. Eliminating the common terms across both integrals yields

$$\begin{aligned} V(\tau+\Delta)-V(\tau) &= \mathbb{1}_{\{T(\tau)<\infty\}} \int_{\tau+\max\{\Delta,T(\tau)\}}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))}u(c(s))ds \\ &\qquad - \int_{\tau}^{\tau+\min\{\Delta,T(\tau)\}} e^{-(\rho+\epsilon)(s-\tau)}u(c(s))ds \\ &\geq \mathbb{1}_{\{T(\tau)<\infty\}} \left(\int_{\tau+\max\{\Delta,T(\tau)\}}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))}ds\right)\underline{u} \\ &\qquad - \left(\int_{\tau}^{\tau+\min\{\Delta,T(\tau)\}} e^{-(\rho+\epsilon)(s-\tau)}ds\right)\overline{u}, \end{aligned}$$

where the inequality follows from the boundedness of u. Solving out the integrals, we obtain

$$\begin{aligned} V(\tau+\Delta)-V(\tau) &= \mathbb{1}_{\{T(\tau)<\infty\}} \left(\frac{e^{(\rho+\epsilon)(\Delta-\max\{\Delta,T(\tau)\})}-e^{-(\rho+\epsilon)T(\tau)}}{\rho+\epsilon}\right)\underline{u} \\ &\qquad - \left(\frac{1-e^{-(\rho+\epsilon)\min\{\Delta,T(\tau)\}}}{\rho+\epsilon}\right)\overline{u} \\ &\geq -\sup_{T\geq 0}\left|\frac{e^{(\rho+\epsilon)(\Delta-\max\{\Delta,T\})}-e^{-(\rho+\epsilon)T}}{\rho+\epsilon}\right||\underline{u}| - \sup_{T\geq 0}\left|\frac{1-e^{-(\rho+\epsilon)\min\{\Delta,T\}}}{\rho+\epsilon}\right||\overline{u}| \\ &= -\left|\frac{1-e^{-(\rho+\epsilon)\Delta}}{\rho+\epsilon}\right|(|\overline{u}|+|\underline{u}|) > -\infty. \end{aligned}$$

A similar argument provides the same lower bound for $V(\tau)-V(\tau+\Delta)$, and thus

$$|V(\tau+\Delta)-V(\tau)| \leq \left|\frac{1-e^{-(\rho+\epsilon)\Delta}}{\rho+\epsilon}\right|(|\overline{u}|+|\underline{u}|).$$

Given that the right-hand side is continuous and goes to zero as $\Delta \to 0$, this guarantees the continuity of $V(\tau)$.

2) $V(\tau) \geq V(0)$ **for all** $\tau \geq 0$**.**

If $V(\tau) < V(0)$ for some $\tau \geq 0$, $F_\tau(\tau) < 1$. A deviation setting $F_\tau(\tau) = 1$ then sets the deviation value equal to $V(0)$.

3) For all $\tau \geq 0$**, there exists** $t \geq \tau$ **such that** $V(t) = V(0)$**.**

We first show for all $\tau \geq 0$, there exists t such that $F_\tau(t) > 0$. Toward a contradiction, suppose there exists $\tau \geq 0$ such that for all $t \geq \tau$, $F_\tau(t) = 0$. This implies for all $t \geq \tau$ that $q(t) = 1$.

Let us now argue that $c(t) < y$ for some $t \geq \tau$. Suppose not and that $c(t) \geq y$ for all $t \geq \tau$. If $b(\tau) > 0$, then this ensures that $b(t)$ eventually exceeds its upper bound, a contradiction of Assumption 2. If $b(\tau) = 0$, $b(t)$ again eventually exceeds its upper bound since $b'(\tau) = H(0, 1) > 0$ from Assumption 3. Thus, there must exist $s \geq \tau$ such that $c(s) < y$.

Assumptions 1 and 2 (and $q(t) = 1$ for all $t \geq \tau$) ensure that $b(t)$ is weakly increasing on $[\tau, \infty)$ and thus $c(t)$ is weakly decreasing on $[\tau, \infty)$. Thus, $c(t) < y$ for all $t \geq s$, ensuring $V(s) < \frac{u(y)}{r+\epsilon}$, contradicting Lemma 1. And thus, there exists t such that $F_\tau(t) > 0$.

Next, define $s \equiv \inf\{t \geq \tau | F_\tau(t) > 0\}$. The previous result guarantees that such an s is finite. Suppose that $V(s) > V(0)$. The continuity of V implies there exists $\Delta > 0$ such that $V(t) > V(0)$ for all $t \in [s, s + \Delta]$. Optimization then implies $F_s(s + \Delta) = 0$. From the definition of s, $F_\tau(s^-) = 0$. Therefore, since $(1 - F_\tau(s + \Delta)) = (1 - F_\tau(s^-))(1 - F_s(s + \Delta))$, $F_\tau(s + \Delta) = 0$. This contradicts the definition of s. And thus, $V(s) = V(0)$.

4) For all $\tau \geq 0$, $V(\tau) = V(0)$.

Steps 1) through 3) establish that if $V(\tau) > V(0)$ for any $\tau > 0$, there must be an open interval containing τ, $(s, s + \Delta)$, such that $V(s) = V(s + \Delta) = V(0)$ and $V(t) > V(0)$ for all $t \in (s, s + \Delta)$. (That is, the function V must have a continuous "hill" containing τ.) Since $V(t) > V(0)$ for all $t \in (s, s + \Delta)$, optimization implies $F_m(n) = 0$ for all $(m, n) \in (s, s + \Delta)^2$ with $n \geq m$. This implies (for $t \in (s, s + \Delta)$) that

$$q(t) = \int_t^{s+\Delta} e^{-(i+\lambda)(m-t)}(i + \lambda)dm + e^{-(i+\lambda)(s+\Delta-t)}q(s + \Delta) = 1 - e^{-(i+\lambda)(s+\Delta-t)}(1 - q(s + \Delta)).$$

Since $q(s + \Delta) < 1$ (from the positive probability of default at some time), $q(t)$ is a continuous strictly decreasing function on $(s, s + \Delta)$.

Next we show that $b(\tau)$ is weakly increasing on $(s, s + \Delta)$. First suppose there exist points $(m, n) \in (s, s + \Delta)^2$ with $m < n$ such that $b'(m) < 0$ and $b'(n) > 0$. (Debt goes down and then up again.) The continuity of b then implies there must exist $(\overline{m}, \overline{n}) \in (s, s + \Delta)^2$ with $m \leq \overline{m} < \overline{n} \leq n$ such that $b(\overline{m}) = b(\overline{n})$ with $b'(\overline{m}) < 0$ and $b'(\overline{n}) > 0$. Assumption 1 then implies $b'(\overline{m}) \geq b'(\overline{n})$ since $q(\overline{n}) < q(\overline{m})$, a contradiction. Thus, if there exists point $m \in (s, s + \Delta)$ such that $b'(m) < 0$, then $b'(n) \leq 0$ for all $n > m$ such that $n \in (s, s + \Delta)$ (or if debt is ever strictly decreasing, it must from then on be weakly decreasing in the open interval). This implies that if $b'(m) < 0$, then $c(n) < y$ for all $n \in [m, s + \Delta)$. But note for all $m \in (s, s + \Delta)$

$$V(m) = \int_m^{s+\Delta} e^{-(r+\epsilon)(t-m)}[u(c(t)) - (r + \epsilon)V(0)]dt + V(0). \tag{33}$$

If $c(t) < y$ for all $t \in [m, s + \Delta)$, then $[u(c(t)) - (r + \epsilon)V(0)] < 0$ for all such t. This contradicts

$V(m) > V(0)$. Thus, $b'(t) \geq 0$ for all $t \in (s, s+\Delta)$, or $b(\tau)$ is weakly increasing on the open interval.

Finally, that $b(\tau)$ is weakly increasing and $q(\tau)$ is strictly decreasing implies that $c(\tau)$ is weakly decreasing on $(s, s+\Delta)$. To see this, note $C_b(b,q) = -i - \lambda(1-q) + qH_b(b,q) < 0$ since $q \leq 1$ and $H_b(b,q) \leq 0$, and $C_q(b,q) = H(b,q) + \lambda b + H_q(q,b) \geq 0$ since $b'(\tau) = H(b(\tau), q(\tau)) \geq 0$ for $\tau \in (s, s+\Delta)$ and $H_q(q,b) \geq 0$ by assumption. We know

$$V(s) = V(0) = \int_s^{s+\Delta} e^{-(r+\epsilon)(t-s)}[u(c(t)) - (r+\epsilon)V(0)]dt + V(0), \tag{34}$$

which implies $\int_s^{s+\Delta} e^{-(r+\epsilon)(t-s)}[u(c(t)) - (r+\epsilon)V(0)]dt = 0$. But that $V(\tau) > V(0)$ for any $\tau \in (s, s+\delta)$, implies that $\int_\tau^{s+\Delta} e^{-(r+\epsilon)(t-\tau)}[u(c(t)) - (r+\epsilon)V(0)]dt > 0$ and $\int_s^{\tau} e^{-(r+\epsilon)(t-\tau)}[u(c(t)) - (r+\epsilon)V(0)]dt < 0$, as their sum is zero. But these mean that there is a $t_0 \in (s,\tau)$ such that $u(c(t_0)) < (r+\epsilon)V(0)$ and there is a $t_1 \in (\tau, s+\Delta)$ such that $u(c(t_1)) > (r+\epsilon)V(0)$. This contradicts $c(\tau)$ weakly decreasing on $(s, s+\Delta)$. Thus, there cannot be a $\tau > 0$ such that $V(\tau) > V(0)$. □

E Proof of Lemma 3

Proof. The functional form of H implies that $C(b,q) = r^*q(\frac{y}{i} - b)$, with $C_b(b,q) = -r^*q$, and $C_q(b,q) = r^*(\frac{y}{i} - b)$. The differential equation in q used to calculate an equilibrium is $q'(\tau) = \frac{C_b(b(\tau),q(\tau))}{C_q(b(\tau),q(\tau))}H(b(\tau),q(\tau))$ then becomes

$$q'(\tau) = \frac{r^*q(\tau)}{r^*(\frac{y}{i} - b(\tau))}\left(\frac{y}{i} - b(\tau)\right)\left(r^* - \frac{i}{q(\tau)}\right) = r^*q(\tau) - i,$$

which is linear in $q(\tau)$. This, with boundary condition $q(T) = \frac{i}{i+\delta}$, allows us to solve this differential equation, delivering

$$q(\tau) = \frac{i}{r^*}(1 + \frac{r^* - i - \delta}{i+\delta}e^{-r^*(T-\tau)})$$

for all $\tau \leq T$ (although at this point, we have not yet derived T).

To derive T, we know (from $\lambda = 0$ and) that $q'(\tau) = (i + x(\tau))q(\tau) - i$, which then, using $q'(\tau) = r^*q(\tau) - i$, implies $x(\tau) = r^* - i$ (or the unconditional arrival rate of default is constant). Applying this to Bayes' rule then implies

$$\rho'(\tau) = (1-\rho(\tau))\epsilon + \rho(\tau)(r^* - i - \delta),$$

which is linear in $\rho(\tau)$, allowing us to solve this differential equation (using the boundary condition $\rho(0) = 0$) as

$$\rho(\tau) = \frac{(e^{(r^*-i-\delta-\epsilon)\tau} - 1)\epsilon}{r^* - i - \delta - \epsilon},$$

for all $\tau \leq T$. Since $\rho(T) = 1$, this in turn allows us to derive T as

$$T = \frac{\log(\frac{r^*-i-\delta}{\epsilon})}{r^* - i - \delta - \epsilon}.$$

Thus, we have closed-form solutions for the evolution of the bond price $q(\tau)$ and reputation $\rho(\tau)$ from $\tau = 0$ to $\tau = T$ (with $\rho(\tau) = 1$ and $q(\tau) = \frac{i}{i+\delta}$ for $\tau \geq T$). Further, since $(1-\rho(\tau))F'_\tau(\tau) = r^* - i$, this gives a closed-form solution for $F'_\tau(\tau)$ (again for $\tau \leq T$) as

$$F'_\tau(\tau) = \frac{(r^* - i)(r^* - i - \delta - \epsilon)}{r^* - i - \delta - e^{\tau(r-i-\delta-\epsilon)}\epsilon}.$$

This leaves us only to solve the differential equation in b, $b'(\tau) = (r^* - \frac{i}{q(\tau)})(\frac{y}{i} - b(\tau))$. Substituting our closed-form solution for $q(\tau)$ when $\tau \leq T$ and solving this differential equation delivers (for $\tau \leq T$)

$$b(\tau) = \frac{(e^{-r^*T} - e^{-r^*(T-\tau)})(r^* - i - \delta)}{(1 - e^{-r^*(T-\tau)})(r^* - i - \delta) - r^*} \frac{y}{i}.$$

For $\tau \geq T$, the same differential equation $b'(\tau) = (r^* - \frac{i}{q(\tau)})(\frac{y}{i} - b(\tau))$ applies, but $q(\tau)$ is constant at $q(\tau) = \frac{i}{i+\delta}$, and the appropriate boundary condition is that debt b at $\tau = T$ is that derived from the closed-form solution for $b(\tau)$ for $\tau \leq T$. Solving the differential equation for this q and boundary condition delivers

$$b(\tau) = (1 - e^{-(r^*-i-\delta)(\tau-T)})\frac{y}{i} + e^{-(r^*-i-\delta)(\tau-T)}b(T).$$

□