

A Contraction for Eaton-Gersovitz*

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Abstract

Using a dual representation, we show that the Markov equilibria of the one-period-bond [Eaton and Gersovitz \(1981\)](#) incomplete markets sovereign debt model can be represented as a fixed point of a contraction mapping, providing a new proof of the uniqueness of equilibrium in the benchmark sovereign debt model. The arguments can be extended to incorporate re-entry probabilities after default when the shock process is *iid*. Our representation of the equilibrium bears many similarities to an optimal contracting problem. We use this to argue that commitment to budget rules has no value to a benevolent government. We show how the introduction of long-term bonds breaks the link to the constrained planning problem.

1 Introduction

This paper provides a compact characterization of the canonical [Eaton and Gersovitz \(1981\)](#) model of sovereign debt with one-period defaultable, but otherwise noncontingent, bonds. In particular, we show that the Markov equilibrium allocation can be characterized in a single Bellman equation that is the fixed point of a contraction mapping. This immediately yields existence and uniqueness, providing an alternative approach to the recent results of [Auclert and Rognlie \(2016\)](#).

The fact that the equilibrium is the solution to a Bellman equation also sheds light on the economics of the equilibrium. The dynamic programming problem shares many similarities with an optimal contracting problem between a principal and an agent that is risk averse and lacks commitment. In this sense, the equilibrium allocation has a number of efficiency properties. However, the contracting problem includes a constraint that reflects market incompleteness, highlighting the inefficiencies that arise due to the ad-hoc constraints on contracts imposed in the Eaton-Gersovitz tradition.

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The argument has two crucial steps. The first is to consider a dual representation of the equilibrium. Specifically, we consider maximizing the value of debt subject to delivering a level of utility to the government. This yields a function $B(s, v)$, which specifies the value of debt when the exogenous state is s and the “promised utility” of the government is v . The choice variables in this program are government consumption (or net payments to the lender) in the current state as well as continuation values in successor periods, $v(s')$ for all s' in some state space S . The optimization is subject to the government’s lack of commitment.

At this point, the description is a textbook principal-agent problem with complete markets and one-sided limited commitment. Hence, the second step of the argument is to ensure that the optimal allocation is consistent with incomplete markets. In particular, the value of debt next period cannot vary across state realizations in which the government repays. This requires that $B(s', v(s'))$ is invariant across all $s' \in S$ in which the government repays. This is reflected in an additional constraint that features the representative lender’s value function.

The equilibrium is then characterized by an operator that maps the space of possible debt values, B , into itself. Differently from a primal approach, our dual representation allows us to show that the function B is the fixed point of a contraction operator. This immediately implies existence and uniqueness. Uniqueness has recently been established by [Auclert and Rognlie \(2016\)](#), who provide a clever proof of uniqueness based on a replication argument similar to [Bulow and Rogoff \(1989b\)](#). One contribution of this paper is to establish uniqueness using standard recursive techniques that are taught in first-year graduate courses.

As a consequence of the dual-contracting approach, the analysis highlights certain efficiency properties of the equilibrium. The ability to commit to future debt issuances cannot improve upon equilibrium allocation, and thus fiscal rules have no value. In particular, aside from the default decision, there is no loss or gain if the government, as part of the contract, cedes future fiscal deficit choices to the lenders. There is also no coordination failure that can lead to a sub-optimal outcome, highlighting the difference between the Eaton-Gersovitz model and the closely related model of [Cole and Kehoe \(2000\)](#) that does feature self-fulfilling crises. The different timing assumptions in the Cole-Kehoe model are not consistent with our dual formulation.

The efficiency properties of short-term bonds stand in contrast to models with long-term bonds.¹ This distinction is made transparent in our dual formulation. The relevant objective for the representative lender is the *market value* of debt. The relevant constraint from incomplete markets is that the *face value* of debt is noncontingent. Conditional on nondefault, with one-period bonds, the market value equals the face value at the time of repayment; hence, the same function B characterizes both the value and the constraint set, and the program searches for a

¹See, for example, [Chatterjee and Eyigungor \(2012\)](#), [Hatchondo and Martinez \(2009\)](#), and [Aguiar, Amador, Hopenhayn, and Werning \(forthcoming\)](#).

single object that satisfies the dual Bellman equation. With long-term bonds, this is no longer the case. This provides a stark reflection of the fact that long-term bonds are subject to dilution (that is, disagreement about fiscal policy between the lender and government) as well as multiplicity (see, for example, [Aguiar and Amador \(2018\)](#) and [Stangebye \(2018\)](#)).

The environment we study hews closely to the canonical one-period Eaton-Gersovitz models popular in the quantitative literature, such as [Aguiar and Gopinath \(2006\)](#) and [Arellano \(2008\)](#). Following the original Eaton-Gersovitz paper, we assume there is no reentry after default, but do allow for arbitrary additional punishments. This makes the deviation utilities primitives of the environment rather than equilibrium objects. The results can be extended to allow for reentry under *iid* endowment shocks, but we do not have results for general Markov processes. As highlighted by, for example, [Passadore and Xandri \(2018\)](#), multiplicity can arise in the Eaton-Gersovitz model if the government is prevented from holding assets. We therefore place no bounds on the space of assets. To ensure that B is bounded, we restrict that consumption lies in an arbitrarily large but compact set; this places a de facto upper bound on the government's value achieved in any equilibrium and hence ensures our equilibrium operator maps bounded functions into bounded functions. As is standard, the government has time-consistent preferences. Therefore, the analysis does not extend to models in which the government has quasi-geometric preferences, such as [Aguiar and Amador \(2011\)](#) or [Alfaro and Kanczuk \(2017\)](#).

The paper is organized as follows. Section 2 introduces the environment and provides a basic characterization of equilibria. Section 3 shows that the equilibrium is a fixed point of a contraction mapping. Section 4 discusses the efficiency of the unique equilibrium and why fiscal rules are not useful. We also discuss why introducing long-term bonds breaks the usefulness of our dual approach. Section 5 shows how the analysis can be extended to the case of reentry after default (as long as the shock process is *iid*). Section 6 concludes. Appendix A collects all of the remaining proofs not included in the main text.

2 The Eaton-Gersovitz Model

Environment. Let us consider the standard sovereign debt model with one-period bonds, originally introduced in [Eaton and Gersovitz \(1981\)](#). Time is discrete, indexed by $t = 0, 1, \dots$, and there is a single tradable good that is the numeraire. There is a small open economy with a government that makes all consumption, debt-issuance, and default decisions. There is also an international financial market, populated by risk-neutral investors that demand a gross interest rate of $R > 1$.

Let s_t denote the exogenous state vector of the economy at time t , and let s^t denote its history, up to and including period t . The exogenous state vector is $s_t \in \mathbb{S}$ where \mathbb{S} is a discrete set. Let

$\pi(s'|s)$ denote the probability of state s' next period given s this period.

The timing within a period is as follows. The small open economy starts with an inherited debt level b , all of which matures in the current period. Nature then draws the current state s . After observing s , the government decides whether to repay its maturing debt or default. If it does not default, the government receives an endowment $y(s)$, auctions new bonds, pays off the maturing debt, and consumes. We restrict attention to Markov-perfect equilibria of the game between the government and the international financial markets.

We impose the following assumption on technology:

Assumption 1. *The transition probabilities are such that $\pi(s'|s) > 0$ for all $(s, s') \in \mathbb{S} \times \mathbb{S}$. In addition, $\min_{s \in \mathbb{S}} y(s) \equiv \underline{y} > 0$ and $\max_{s \in \mathbb{S}} y(s) \equiv \bar{y} < \infty$.*

Let $V^R(s, b)$ denote the equilibrium value to the government if it chooses to repay its debt given (s, b) . If it defaults, we assume that the payoff to the government is exogenous and equal to $V^D(s)$. Upon default, we assume that the lenders receive a payoff of zero. The default payoff for the government may depend on the exogenous state, but does not vary with the amount of debt at the time of default. Importantly, in this formulation, the default payoff is a primitive and does not depend on the other aspects of the equilibrium.² Strategic default implies that the government defaults if $V^R(s, b) < V^D(s)$. We assume, as it is standard, that the government repays when indifferent, that is, when $V^R(s, b) = V^D(s)$.

Government Optimization. Let us now consider the government's problem. To rule out Ponzi schemes, we impose that $b \leq \bar{B}$, where we assume that \bar{B} exceeds the present value of the maximal endowment: $\bar{B} > \bar{y}R/(R - 1)$. We let $u(c)$ denote the utility flows received by the government given an associated expenditure level c , and we assume that the government evaluates alternative spending plans discounting future expected utility flows with an exponential factor $\beta < 1$.

We assume consumption is chosen from a compact set: $[0, \bar{c}]$. As will become clear below, an upper bound on consumption allows us to focus on a finite threshold for assets in any equilibrium.³ The bound on consumption can be set to any arbitrarily large finite number, and in particular we make the natural assumption that it is always possible to consume the present value of the endowment in any period:

Assumption 2. $\bar{c} > R\bar{y}/(R - 1)$.

Define the upper bound on the value function by $\bar{V} \equiv u(\bar{c})/(1 - \beta)$.

²This is consistent with autarky and exogenous output losses as punishments for default, but it does not admit reentry to financial markets, an extension that we discuss later on.

³This assumption helps guarantee that the dual operator we provide below maps bounded function into bounded functions.

We assume that the government always prefers to default rather than consume zero in the current period:

Assumption 3. *There exists a $\underline{c} \in (0, \bar{c})$ such that*

$$u(\underline{c}) + \beta \bar{V} < \min_{s \in \mathbb{S}} V^D(s). \quad (1)$$

This last assumption implies that, conditional on repayment, consumption is bounded away from zero — a feature that is helpful when proving the strict monotonicity of the equilibrium value function.

We now discuss the government's problem conditional on repayment. The government faces an equilibrium price schedule q that maps the current state and newly issued debt $b' \in \mathbb{R}$ into $[0, R^{-1}]$, where R is the world risk-free rate. Given the fact that the government commits to repaying maturing debt b prior to auctioning b' , q is not a function of b .⁴ Hence, the budget constraint of the government in state (s, b) , if the government decides to repay, is $c \leq y(s) - b + q(s, b')b'$.

Let B^F denote the maximal debt that is feasible to repay in state s :

Definition 1. *In any equilibrium, $B^F(s)$ represents the largest amount of debt that is feasible to repay in state s :*

$$B^F(s) \equiv y(s) + \sup_{b' \leq \bar{B}} q(s, b')b'. \quad (2)$$

Note that B^F depends on q and hence is an equilibrium outcome. If the current amount of debt due, b , is such that $b > B^F(s)$, the government has no alternative to default as repayment is not feasible. In that case, we let $V^R(s, b) = V_{NF}$ if $b \geq B^F(s)$, for some sufficiently low value $V_{NF}(s) < \min_{s \in \mathbb{S}} V^D(s)$.

Given exogenous state s and inherited debt b , the government's problem *conditional on repay-*

⁴See Cole and Kehoe (2000), and the discussion in Aguiar and Amador (2014), for the implications of an alternative timing.

ment can be written recursively as follows:

$$\text{If } b < B^F(s): V^R(s, b) = \sup_{c \in [0, \bar{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \{V^R(s', b'), V^D(s')\} \right\} \quad (\text{G})$$

subject to:

$$\begin{aligned} c &\leq y(s) - b + q(s, b')b', \\ b' &\leq \bar{B}. \end{aligned}$$

$$\text{If } b \geq B^F(s): V^R(s, b) = V_{NF}$$

Note that the specific value of V_{NF} does not affect the value function $V^R(s, b)$ for $b < B^F(s)$, as such value is off-equilibrium (that is, it always triggers default).

Lenders' Break-even Condition. Given the risk neutrality of lenders (and infinite collective wealth), we replace the lenders' problem and market clearing in the international financial markets with a break-even condition. In particular, given (s, b) and conditional on the government repaying and auctioning b' , let $q(s, b')$ denote the equilibrium price of a bond that pays one next period absent default. The lenders' break-even condition for any $s \in \mathbb{S}$ and $b' \leq \bar{B}$ is⁵

$$q(s, b') = \begin{cases} R^{-1} & \text{if } b' \leq 0 \\ R^{-1} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b') \geq V^D(s')\}} & \text{otherwise,} \end{cases} \quad (\text{BE})$$

where R^{-1} denotes the international discount factor, as described previously. The first row on the right-hand side states that (positive) assets return the risk-free rate.⁶ The second row reflects that debt will be repaid only if doing so is feasible and optimal for the government. A crucial feature of (BE) is that current prices depend on the future only through the associated continuation value V^R ; in particular, the debt policy function that generates $V^R(s, b')$ is not directly relevant. This is a consequence of one-period debt.

The Value with No Access to Debt. Before moving on to characterize the equilibrium with borrowing, let us consider a restricted problem, where the government is only allowed to save and cannot borrow, that is, $b \leq 0$. Let $V^{NA}(s, b)$ denote the corresponding government's value

⁵Here, $\mathbb{1}_x$ denotes the indicator function that takes one if x is true and zero otherwise.

⁶We have not explicitly ruled out default when the government holds assets abroad. However, Lemma 1 shows that our Assumption 4 below is sufficient for this.

function. This value function is the *unique* bounded fixed point of the following Bellman: equation

$$V^{NA}(s, b) = \max_{c \in [0, \bar{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V^{NA}(s', b') \right\} \quad (\text{NA})$$

subject to:

$$\begin{aligned} c &\leq y(s) - b + R^{-1}b' \\ b' &\leq 0 \end{aligned}$$

We impose the following assumption, which implies that default must entail some cost:

Assumption 4. For all $s \in S$, $V^{NA}(s, 0) \geq V^D(s)$.

As established by [Auclert and Rognlie \(2016\)](#), Assumption 4 implies that the government will never default without strictly positive debt (a result we confirm below).

Definition of Equilibrium. Let $\mathbb{X} \equiv \{(s, b) \text{ such that } s \in S \text{ and } b \leq \bar{B}\}$, the set of admissible states. With this, we can now define an equilibrium:

Definition 2. A Markov-perfect equilibrium consists of two functions, $V^R : \mathbb{X} \rightarrow \mathbb{R}$ and $q : \mathbb{X} \rightarrow [0, 1/R]$ such that: (i) given q , V^R satisfies (G) for (s, b) in \mathbb{X} , and the supremum in (G) is attained for some policy; and (ii) given V^R , q satisfies (BE).

Characterizing Equilibria

We now establish key properties of the government's equilibrium value function. In particular, we establish that the value function is continuous, weakly decreasing, and strictly decreasing whenever \bar{V} is not feasible, as depicted in Figure 1.

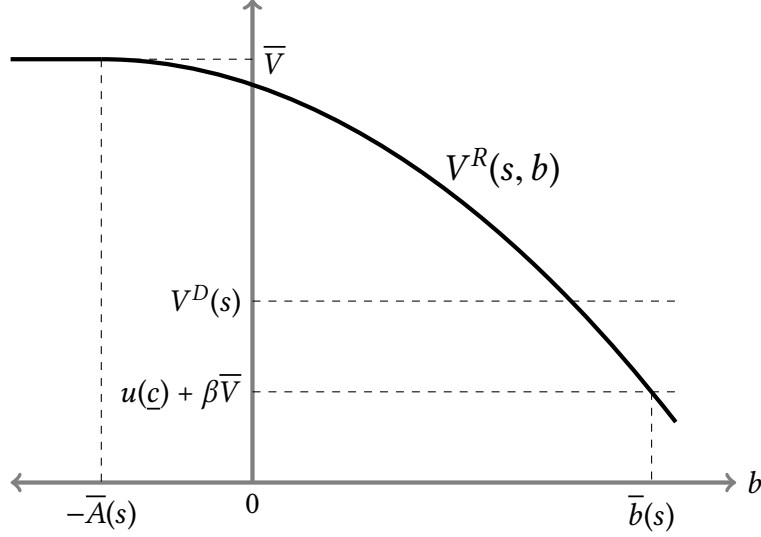
Preliminaries

We first establish some preliminary results. The first is a property of the feasible repayment set, $B^F(s)$. Using Definition 1 and that $q(s, b')b' \leq R^{-1}\bar{B}$, we have

$$B^F(s) \leq \bar{y} + R^{-1}\bar{B} < \bar{B}, \text{ for all } s \in \mathbb{S}.$$

That is, government borrowing is bounded by the present value of the endowment path that features the highest income realization forever.

Figure 1: The Government's Value Function



Note: The diagram depicts $V^R(s, b)$ for fixed $s \in \mathbb{S}$ as a function of b . Thresholds $-\bar{A}(s)$ and $\bar{b}(s)$ are defined in Definition 3 and Lemma 3, respectively. Consumption level \underline{c} is defined in Assumption 3. Note that Assumption 3 implies that $V^D(s)$ is strictly above $u(\underline{c}) + \beta \bar{V}$. Thus, default occurs in state s for levels of debt below $\bar{b}(s)$.

The next lemma states that it is never optimal to default with a weakly positive net asset position (Part (i)),⁷ and it is never optimal to issue debt at a zero price (Part (ii)):

Lemma 1. *In any equilibrium,*

- (i) *For any state $s \in \mathbb{S}$ and $b \leq 0$, $V^R(s, b) \geq V^D(s)$; and*
- (ii) *For any state $s \in \mathbb{S}$ and $b < B^F(s)$, there exists an optimal debt choice b' and at least one element $s' \in \mathbb{S}$ such that $V^R(s', b') \geq V^D(s')$.*

Part (ii) implies that there is always an optimal debt policy such that the government never defaults with probability one next period.

■ *Proof.* The proof in the appendix. □

The next set of results concern the feasibility of the maximal consumption, \bar{c} , and the maximal government value, \bar{V} . Let us define a level of assets that is sufficient to finance \bar{c} forever regardless of future endowment realizations:

⁷Hence, setting $q(s, b') = R^{-1}$ for $b' \leq 0$ is without loss given Assumption 4.

Definition 3. Let $\bar{A}(s)$ be such that

$$\bar{A}(s) \equiv \bar{c} - y(s) + \frac{\bar{c} - y}{R - 1} \text{ for all } s \in \mathbb{S}. \quad (3)$$

Note that $\bar{A}(s) > 0$. We have:

Lemma 2. In any equilibrium, for any state $s \in S$, $V^R(s, b) = \bar{V}$ if and only if $b \leq -\bar{A}(s)$. Moreover, if $c = \bar{c}$ for any state $(s, b) \in \mathbb{X}$, then $b < 0$.

■ *Proof.* The proof in the appendix. □

Properties of the Value Function

The following establishes continuity and weak monotonicity of V^R on the relevant domain for debt:

Lemma 3. In any equilibrium,

(i) For any $s \in \mathbb{S}$, $V^R(s, b)$ is weakly decreasing for $b < B^F(s)$.

(ii) For any $s \in \mathbb{S}$, there exists a unique threshold $\bar{b}(s) < B^F(s)$ such that:

$$V^R(s, \bar{b}(s)) = u(\underline{c}) + \beta \bar{V}.$$

(iii) For any $s \in \mathbb{S}$, $V^R(s, b)$ is continuous for $b \leq \bar{b}(s)$.

■ *Proof.* The proof in the appendix. □

We can now strengthen the monotonicity result:

Lemma 4. In any equilibrium, for all $s \in \mathbb{S}$, $V^R(s, b)$ is strictly decreasing in b for $b \in (-\bar{A}(s), \bar{b}(s)]$.

■ *Proof.* The proof is in the appendix. □

3 An Eaton-Gersovitz Contraction Operator

In this section, we proceed to show that the value function V^R must solve a dual problem, whose solution can be represented as the fixed point of a contraction mapping.

Toward this goal, we first combine the government's problem (G) with the lenders' break-even

constraint (BE) to write the equilibrium problem as

$$V^R(s, b) = \max_{c \in [0, \bar{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \{ V^R(s', b'), V^D(s') \} \right\} \quad (\text{G}')$$

subject to:

$$c \leq y(s) - b + b' R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b') \geq V^D(s')\}} \right],$$

$$b' \leq \bar{B},$$

where again, we let $V^R(s, b) = V_{NF}$ if the constraint set is empty.

Problem (G') has the familiar recursive structure that takes a continuation value function and maps it into the current value state by state. Any equilibrium value function V^R is a fixed point of the operator defined by this Bellman equation. The quantitative sovereign debt literature has developed algorithms to find this fixed point numerically. While the operator is monotone and maps the space of bounded functions into itself (if u is bounded), it is not clear that it satisfies discounting. As a result, one cannot appeal to Blackwell's sufficient conditions to establish that the operator is a contraction. The difficulty lies in the complicated manner in which V^R appears in the constraint set.

Fortunately, we show below that there is a transformation that delivers an operator that does satisfy all of Blackwell's sufficient conditions. This alternative operator involves the dual problem to (G').

3.1 The Dual Problem

Given an equilibrium value $V^R(s, b)$, let us define the dual of the optimization problem in (G'):

$$\hat{B}(s, v) \equiv \sup_{c \in [0, \bar{c}], b'} \left\{ y(s) - c + R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b') \geq V^D(s')\}} \right] b' \right\} \quad (\text{B})$$

subject to:

$$v = u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \{ V^R(s', b'), V^D(s') \}, \quad (4)$$

$$b' \leq \bar{B}. \quad (5)$$

We now establish a basic duality result, namely, that the inverse of the government's value function satisfies problem (B). Specifically, Lemmas 2, 3, and 4 imply that in any equilibrium,

there exists a continuous, strictly decreasing function $B(s, v)$ such that

$$v = V^R(s, B(s, v))$$

for all $s, v \in \mathbb{S} \times \mathbb{V}$ where $\mathbb{V} \equiv [u(\underline{c}) + \beta\bar{V}, \bar{V}]$, with $B(s, \bar{V}) = -\bar{A}(s)$ and $B(s, u(\underline{c}) + \beta\bar{V}) = \bar{b}(s)$ for all $s \in \mathbb{S}$. That is, $B(s, v)$ is the inverse of equilibrium value function $V^R(s, b)$ with respect to its second argument, b , over its strictly decreasing range $(-\bar{A}(s), \bar{b}(s))$. Then, we have the following duality result:

Lemma 5. *The function $B(s, v) = \hat{B}(s, v)$ for all $(s, v) \in \mathbb{S} \times \mathbb{V}$.*

■ *Proof.* The proof is in the appendix. □

In the next subsection, we show how the solution to problem (B) can be represented as the fixed point of a contraction mapping operator.

3.2 The Equilibrium Operator

The inverse value function B is a fixed point of an operator implicitly defined in (B). More formally, define the following operator T on functions $f : \mathbb{S} \times \mathbb{V} \rightarrow \mathbb{R}$:

$$Tf(s, v) = \sup_{c \in [0, \bar{c}], b', \{w(s')\}_{s' \in \mathbb{S}}} \left\{ y(s) - c + R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{w(s') \geq V^D(s')\}} \right] b' \right\} \quad (\text{T})$$

subject to:

$$v \leq u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \{w(s'), V^D(s')\} \quad (6)$$

$$b' \leq f(s', w(s')) \text{ for all } s' \in \mathbb{S} \text{ such that } w(s') \geq V^D(s') \quad (7)$$

$$w(s') \in \mathbb{V} \text{ for all } s' \in \mathbb{S}.$$

Before discussing useful properties of this operator, we discuss differences with the original dual problem (B). The key alteration is that the government's repayment value V^R no longer appears in the problem. Rather, the problem allows the choice of the government's continuation value state by state, represented by $\{w(s')\}_{s' \in \mathbb{S}}$. In this sense, the problem shares a passing resemblance to a standard contracting problem in which a risk-neutral principal insures a risk-averse agent, subject to limited commitment on the part of the agent. We shall return to this point in Section 4.

However, recall that one crucial friction in the Eaton-Gersovitz model is the lack of state-contingent liabilities. This is accommodated by the presence of b' in the objective and the con-

straint (7). Specifically, the continuation value in the objective is a scalar, b' , rather than a state-contingent vector of values. This is the noncontingent debt carried into the next period.

Moreover, the choice of the government's continuation values must be consistent with the choice of b' . Hence, a new constraint (an *implementability constraint*) is introduced in (7). The equilibrium imposes that debt and the government's payoffs be related by $b' = B(s', w(s'))$ for all s' , which is equivalent to $w(s') = V^R(s', b')$ for all s . In problem (T), we have relaxed this equality constraint to an inequality, imposing it only for levels of the continuation value that do not trigger default. Note that, relative to (B), we have also replaced the equal sign in the promise keeping constraint with an inequality in (6) and dropped the no-Ponzi condition, which imposed $b' \leq \bar{B}$.

Despite these alterations, the equilibrium B that solves (B) is a fixed point of the operator defined by (T):

Lemma 6. *Any equilibrium $B(s, v)$ is a fixed point of T .*

Proof. The proof is in the appendix. □

3.3 A Contraction Mapping

We have shown above that the operator T admits as a fixed point the dual of any equilibrium value function. Interestingly, even though the original operator that defined V^R was not a contraction mapping, the operator T , which works on the dual of V^R , is a contraction mapping. We now proceed to show this.

Toward that goal, we endow the space of functions on which T operates with the sup norm. Our first statement is that T maps bounded functions into bounded functions:

Lemma 7 (Boundedness). *Let $f : \mathbb{S} \times \mathbb{V} \rightarrow \mathbb{R}$ be bounded in the sup norm. Then Tf is a bounded function.*

Proof. Consider a bounded f such that $\|f\| < M$. For any state s_0, v_0 , consider the policy of $c = \bar{c}$ and $w(s') = \bar{V}$ for all $s' \in \mathbb{S}$. Let $b'_0 \leq -M$. The policy $c_0, b'_0, \{w_0(s')\}$ satisfies the constraint set of problem (T), and thus

$$(Tf)(s_0, v_0) \geq y(s_0) - \bar{c} - M/R \geq \underline{y} - \bar{c} - M/R,$$

but also

$$(Tf)(s_0, v_0) \leq \bar{y} + M/R.$$

Thus, $\|(Tf)(s_0, v_0)\| \leq \max\{\bar{y} + M/R, \underline{y} - \bar{c} + M/R\}$, which is independent of (s_0, v_0) , and thus, $\|Tf\| \leq \max\{\bar{y} + M/R, \underline{y} - \bar{c} + M/R\}$, which is bounded. □

Note that this is the place where our assumption that consumption of the government has

an upper bound has really been used — it guarantees that the dual operator T maps bounded functions into bounded functions. Note also that the particular value of \bar{c} is irrelevant for all of the analysis above, as long as it is large enough so that Assumption 2 is satisfied.

We next show that the operator T is monotone, a property also shared with the original operator implicit in (G') :

Lemma 8 (Monotonicity). *Let f, g be bounded functions mapping $\mathbb{S} \times \mathbb{V}$ to \mathbb{R} , with $f(s, v) \leq g(s, v)$ for all $(s, v) \in \mathbb{S} \times \mathbb{V}$. Then, $(Tf)(s, v) \leq (Tg)(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$.*

Proof. Note that $(Tg)(s_0, v_0)$ only differs from $(Tf)(s_0, v_0)$ because of constraint (7). It follows that any choice available at $(Tf)(s_0, v_0)$ is also feasible at $(Tg)(s_0, v_0)$ and delivers the same objective. Hence, $(Tf)(s, v) \leq (Tg)(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$. \square

The final step, and the one where the dual representation is exploited, is to show that the operator T satisfies the discounting property with module R^{-1} :

Lemma 9 (Discounting). *Let $a \geq 0$ and let $f : \mathbb{S} \times \mathbb{V} \rightarrow \mathbb{R}$ be bounded. Then,*

$$[T(f + a)](s, v) \leq (Tf)(s, v) + R^{-1}a$$

for all $s, v \in \mathbb{S} \times \mathbb{V}$.

Proof. Let $a > 0$ and we have

$$[T(f + a)](s, v) = \max_{c, v(s'), b'} \left\{ y(s) - c + R^{-1} \max\{0, b'\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{v(s') \geq V^D(s')\}} + R^{-1} \min\{0, b'\} \right\}$$

subject to:

$$v \leq u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{v(s'), V^D(s')\}$$

$$b' \leq f(s', v(s')) + a \text{ for all } s' \in \mathbb{S} \text{ such that } w(s') \geq V^D(s'),$$

$$c \in [0, \bar{c}]$$

$$w(s') \in \mathbb{V} \text{ for all } s' \in \mathbb{S},$$

where we have just replaced $b' \mathbb{1}_{\{b' > 0\}}$ with $\max\{0, b'\}$ and $b' \mathbb{1}_{\{b' \leq 0\}}$ with $\min\{0, b'\}$.

We can rewrite the final two terms in the objective as

$$\begin{aligned}
& R^{-1} \max\{0, b'\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + R^{-1} b' \min\{0, b'\} = \\
& R^{-1} \max\{-a, b' - a\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + R^{-1} \min\{-a, b' - a\} \\
& + aR^{-1} \left(\sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + 1 \right) \leq \\
& R^{-1} \max\{0, b' - a\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + R^{-1} \min\{0, b' - a\} \\
& + aR^{-1} \left(\mathbb{I}_{\{b' > a\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + \mathbb{I}_{\{b' \leq a\}} \right) \leq \\
& R^{-1} \max\{0, b' - a\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} + R^{-1} \min\{0, b' - a\} \\
& + aR^{-1}.
\end{aligned}$$

Defining $\hat{b} \equiv b' - a$, this implies

$$\begin{aligned}
[T(f + a)](s, v) & \leq \max_{c, v(s'), \hat{b}} y(s) - c + R^{-1} \max\{0, \hat{b}\} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} \\
& + R^{-1} \min\{0, \hat{b}\} + R^{-1} a \\
& \text{subject to:} \\
v & \leq u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{v(s'), V^D(s')\} \\
\hat{b} & \leq f(s', v(s')) \text{ for } s' \in \mathbb{S}.
\end{aligned}$$

Note that this problem is identical to the original, save for $R^{-1}a$ in the objective. In particular, $[T(f + a)](s, v) \leq [Tf](s, v) + R^{-1}a$. Thus, T discounts with modulus R^{-1} . \square

Lemmas 7, 8, and 9 imply that the operator T satisfies Blackwell's sufficient conditions. Hence, the operator T is a contraction with modulus R^{-1} . The contraction mapping theorem states that T has a unique fixed point in the space of bounded functions. Recall that we have shown that $B(s, v)$ is a fixed point of T , and thus there is at most one equilibrium in the Eaton-Gersovitz model:

Corollary 1. *There is at most one Markov-perfect equilibrium.*

Although we do not pursue this here, it is possible to use our contraction mapping argument to prove the existence of an inverse equilibrium value function and, as a result, the existence of a Markov-perfect equilibrium.⁸

3.4 Discussion of Uniqueness

Auclert and Rognlie (2016) is the first proof of uniqueness in the Eaton-Gersovitz model. Au-

⁸Auclert and Rognlie (2016) provide an alternative proof of existence using the monotonicity of the primal operator.

[Auerclert and Rognlie \(2016\)](#) use a different approach to establish the result. In particular, they prove uniqueness by contradiction. Assuming a second equilibrium, the authors construct portfolios that mimic the allocation in the original equilibrium. Thus, the government’s welfare is pinned down by the best equilibrium. As prices depend only on the government’s value, this uniquely determines prices. There is a link to our proof in that we both exploit the fact that prices only depend on the government’s value next period, and not on future fiscal policies. As we shall see, longer maturity debt does not have this feature, and the equilibrium is not necessarily unique.

If the government is restricted from holding assets, then the equilibrium is not unique. [Pasadore and Xandri \(2018\)](#) discuss multiplicity in an environment without assets. If we restrict $b' \geq 0$, we need an additional constraint in (T). In that case, our proof that T satisfies discounting is not valid. [Auerclert and Rognlie \(2016\)](#) show that allowing for an arbitrarily small amount of assets is sufficient to restore uniqueness. The key is that there is some level of assets (or debt) for which default is never optimal regardless of creditor expectations.

On the Irrelevance of Sunspots. Suppose that we were to enlarge the state space \mathbb{S} by including an additional state variable $z \in \mathbb{Z}$, unrelated to any payoff relevant state. This could represent, for example, a sunspot random variable or something related to the history of actions taken by the government. Let $(s, z) \in \mathbb{S} \times \mathbb{Z}$ represent an element of this new state space. The same arguments as above tell us that there is a unique Markov equilibrium, with a value function $V^R((s, z), b)$ that is the unique fixed point of operator (G') under the enlarged state space. Now suppose $V^R(s, b)$ is the fixed point of (G') under the original state space that restricts attention to payoff relevant states $s \in S$. It immediately follows that $V^R((s, z), b) = V^R(s, b)$ is a fixed point under the enlarged state space. Given that the fixed point is unique, there are no other equilibria. Thus, sunspots or payoff irrelevant state variables have no impact on the equilibrium of the model.

Bulow and Rogoff (1989)’s argument In their 1989 paper, [Bulow and Rogoff](#) show that no strictly positive level of debt is sustainable if the government can save after default, and there are no other direct costs of default besides the inability to borrow again. Their argument is based on an arbitrage and, as a result, quite general. It is possible to use the uniqueness of the Markov-perfect equilibrium to show that the same result holds in the [Eaton and Gersovitz \(1981\)](#)’s environment.⁹

Toward this end, let $V^D(s) = V^{NA}(s, 0)$. That is, the government, after default, can save but cannot borrow again. The Bulow-Rogoff claim is that, given this outside option, borrowing is not sustainable in equilibrium. To prove the Bulow-Rogoff claim, we posit that it is true and construct

⁹[Auerclert and Rognlie \(2016\)](#) use their replication argument to show that the [Bulow and Rogoff \(1989b\)](#) holds in the Eaton-Gersovitz model. See also [Bloise, Polemarchakis, and Vailakis \(2017\)](#) for a general argument under incomplete markets.

the associated equilibrium value function. If the associated value is a fixed point of (G') , then zero borrowing is the only possible equilibrium outcome.

Specifically, conjecture the following equilibrium price schedule:

$$q^{BR}(s, b) = \begin{cases} R^{-1} & \text{for } b \leq 0, \\ 0 & \text{for } b > 0, \end{cases}$$

and value function, $V^{BR}(s, b)$, defined for $b \leq y(s)$ as

$$V^{BR}(s, b) = \max_{c \in [0, \bar{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V^{NA}(s, b') \right\}$$

subject to:

$$c \leq y(s) - b + R^{-1}b',$$

$$b' \leq 0,$$

and as $V^{BR}(s, b) = V_{NF}$ for $b > y(s)$ (as before).

Note that $V^{BR}(s, b) = V^{NA}(s, b)$ for $b \leq 0$ and $V^{BR}(s, b) < V^{NA}(s, 0) = V^D(s)$ for $b > 0$ (this last following from strict monotonicity of the problem above). This value function justifies the conjectured price q^{BR} and is a fixed point of (G') . Given that there is only one fixed point of (G') , it follows then that $\{q^{BR}, V^{BR}\}$ is the *unique* Markov equilibrium. This equilibrium entails immediate default for any $b > 0$: no level of borrowing can be sustained.

4 Constrained Efficiency and Why Fiscal Rules Add No Value

In this section, we show that the equilibrium of the [Eaton and Gersovitz \(1981\)](#) model with one-period bonds is *constrained efficient*, when the incompleteness of the markets and the government's inability to commit to repayment are both taken into account.

To understand this point, consider a situation where the government at time $t = 0$ commits to a sequence of debt issuances as a function of the history of shocks: $\mathbf{b} = \{\mathbf{b}(s^t)\}_{t, s^t}$, where $s^t = (s_0, s_1, \dots, s_t)$ denotes the history of exogenous shocks through time t and $\mathbf{b}(s^t)$ is the amount of debt issued at history s^t and due in period $t + 1$. We can think of such a state-contingent debt-issuance policy as arising from a constitutional fiscal rule. The government, however, is still able to default if its equilibrium value of following this rule lies below the corresponding outside option for that state.

The potential value to the government of committing to such a rule is that it potentially affects equilibrium prices. That is, as the government is large in its own debt market, it recognizes that there is a corresponding sequence of equilibrium prices associated with a particular fiscal rule.

Define the sequence $\{c(s^t), v(s^t)\}$ associated with a fiscal rule, given an equilibrium price $\mathbf{q} = \{\mathbf{q}(s^t)\}$, by the following recursion:

$$\begin{aligned} c(s^t) &\equiv \max \{y(s^t) - \mathbf{b}(s^{t-1}) + \mathbf{q}(s^t)\mathbf{b}(s^t), \bar{c}\} \\ v(s^t) &\equiv u(c(s^t)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s^t) \max \{v(\{s^t, s'\}), V^D(s')\}, \end{aligned} \quad (8)$$

where we let $v(s^t) = V^{NF}$ if $c(s^t) < 0$, as before. The price must keep lenders indifferent, and thus,

$$\mathbf{q}(s^t) \equiv \begin{cases} R^{-1} \sum_{s'} \pi(s'|s) \mathbb{1}\{v(\{s^t, s'\}) \geq V^D(s')\} & \text{if } \mathbf{b}(s^t) > 0 \\ R^{-1} & \text{if } \mathbf{b}(s^t) \leq 0. \end{cases} \quad (9)$$

Thus, associated with any fiscal rule is a sequence $\{c(s^t), v(s^t), \mathbf{q}(s^t)\}_{t,s^t}$. The fiscal rule design problem is then to choose \mathbf{b} to maximize initial value, $v(s_0)$ given $\mathbf{b}(s^{-1}) = b_0$:

$$V^*(s_0, b_0) \equiv \sup_{\{\mathbf{b}, \mathbf{q}, \{v(s^t), c(s^t)\}\}} v(s_0) \text{ subject to } \mathbf{b}(s^{-1}) = b_0, \text{ (8) and (9)}. \quad (10)$$

In this fiscal design rule, we are assuming that the designer can choose both the debt sequence and its associated price (as long as the latter satisfies the break-even condition for the lenders). In this way, the designer is allowed to choose the best price (if there were many consistent with a given fiscal rule). That is, the designer can coordinate the lenders' expectations. As we will argue next, there is no value to the fiscal rule even in this case. As a result, there will be no value either when the designer cannot coordinate the lenders' expectations.

Using the dynamic programming principle, it follows that V^* must solve

$$V^*(s_0, b_0) = \sup_{c_0, b_1, q_1, \{\mathbf{b}(s^t), \mathbf{q}(s^t), v(s^t)\}_{t \geq 1}} \left\{ u(c_0) + \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \max \{v(\{s_0, s'\}), V^D(s')\} \right\} \quad (11)$$

such that

$$c_0 = \max \{y(s_0) - b_0 + q_1 b_1, \bar{c}\} \quad (12)$$

$$q_1 = \begin{cases} R^{-1} \sum_{s'} \pi(s'|s) \mathbb{1}\{v(\{s_0, s'\}) \geq V^D(s')\} & \text{if } b_1 > 0 \\ R^{-1} & \text{if } b_1 \leq 0 \end{cases} \quad (13)$$

$$\{\mathbf{v}(s^t), \mathbf{b}(s^t), \mathbf{q}(s^t)\}_{t \geq 1} \text{ satisfy } \mathbf{b}(\{s_0, s'\}) = b_1 \text{ for all } s' \in \mathbb{S}, \text{ (8) and (9)}. \quad (14)$$

Note that it is optimal to choose a continuation sequence $\{v(s^t), \mathbf{b}(s^t), \mathbf{q}(s^t)\}_{t \geq 1}$ such that $v(\{s_0, s'\}) = V^*(s', b_1)$ for all $s' \in \mathbb{S}$. Replacing $v(\{s_0, s'\})$ by $V^*(s', b_1)$ in the above, we have an operator that maps the space of potential V^* into itself. The value associated with the optimal fiscal rule is a fixed point of this operator. This operator is identical to that defined by the equilibrium in problem (G'). Given that we have shown that there is a unique fixed point to this operator, it follows that $V^R(s, b) = V^*(s, b)$. Thus, the ability to commit to a fiscal rule offers no scope to increase the government's value over the Markov-perfect equilibrium value.

A critical feature of the fiscal rule design problem above is that the value of autarky, $V^D(s)$, is not affected by the rule. This is natural under the assumption that, once the country defaults, it cannot access financial markets again, and as a result, it is restricted to consuming its (reduced) endowment. If we were to change the environment and allow the designer to affect, through the fiscal rule, the value of default (and hence, equilibrium prices), then it is possible to construct examples where a fiscal rule generates a value higher than the Markov-perfect equilibrium. It is not surprising that it may be desirable to manipulate the outside option of an agent in this limited commitment model – as this potentially relaxes a main friction in the environment.¹⁰ Our point here is that, beyond this (that is, given the value of default to the government), the equilibrium allocation cannot be further improved once the incompleteness of markets is taken into account.

4.1 Why the Contraction Fails with Long-term Bonds

Let us now briefly extend the model to incorporate long-duration bonds as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). As is now well known, long-duration bonds generate an inefficiency into the environment, a point analyzed in detail in Aguiar et al. (forthcoming).

The environment is modified in the following way. Rather than issuing a one-period bond, the government instead issues a perpetual claim to an exponentially declining coupon. Specifically, a perpetuity issued at time t offers to pay a coupon κ in period $t + 1$, $\kappa(1 - \delta)$ in period $t + 2$, $\kappa(1 - \delta)^2$ in period $t + 3$, and so on. The parameter δ controls the speed at which the coupon decays: $\delta = 1$ corresponds to the one-period bond, and $\delta = 0$ corresponds to a perpetuity that never decays. Define b as the stock of debt entitled to a coupon κ today; hence, absent issuance, b decays at the rate δ .

In a Markov-perfect equilibrium of the long-duration bond model, the government solves the

¹⁰For other examples where a policy that affects the outside/default option of the agent in a limited commitment model is beneficial see Aguiar, Amador, and Gopinath (2009) in a context with investment, and Arellano and Heathcote (2010) in a sovereign default model with dollarization.

following problem:

$$V^R(s, b) = \sup_{\{c \in [0, \bar{c}], b \leq \bar{B}\}} \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) \{V^R(s', b'), V^D(s')\} \right\}$$

subject to $c \leq y - \kappa b + q(s, b')(b' - (1 - \delta)b)$,

where $b' - (1 - \delta)b$ represents the amount of new bond issuances and $q(s, b')(b' - (1 - \delta)b)$ the amount of revenue raised from them. Let $\mathcal{B}(s, b)$ denote an associated equilibrium debt policy function.

Risk-neutral pricing from the perspective of the lenders leads to the following break-even condition:

$$q(s, b') = \begin{cases} R^{-1} & \text{if } b' < 0 \\ R^{-1} \sum_{s'} \pi(s'|s) \mathbb{1}_{[V^R(s', b') \geq V^D(s')]} [\kappa + (1 - \delta)q(s', \mathcal{B}(s', b'))] & \text{if } b' \geq 0. \end{cases}$$

In case of no default next period, the bondholders receive both the coupon as well as the market value of the remaining bond: $\kappa + (1 - \delta)q(s', \mathcal{B}(s', b'))$.

The important element to highlight here is the presence of the equilibrium debt policy function, evaluated at the subsequent state: the price of the long-duration bond depends not only on the debt policy chosen today (b'), but also on the debt policy that the government will choose in subsequent periods. Even if $V^R(s, b)$ is strictly decreasing over some domain, implying that there is a well-defined inverse $B(s, v)$ that maps the government's value to the face value of debt, *this mapping is conditional on a policy \mathcal{B}* . Hence, the equilibrium cannot be written as the fixed point of a contraction mapping, as was the case for the one-period bond model. Indeed, as shown in [Aguiar and Amador \(2018\)](#), there exists parameter values such that the long-duration model features multiple equilibria — each of them featuring different issuance policies.

5 Reentry after Default

In our previous analysis, we have assumed that default entails permanent exclusion from financial markets. The quantitative literature, however, usually assumes that exclusion is a transitory state: a government eventually reaccesses the international financial markets. In this section, we show that, under the assumption that shock process s is *iid* across time, it is possible to extend our dual approach to show uniqueness when reentry subsequent to default is possible.¹¹

Toward this, let $V^D(s)$ denote the value of default under no reentry. The assumption is that

¹¹[Auclert and Rognlie \(2016\)](#) also extend their uniqueness proof to encompass reentry under an *iid* shock process.

as long as the government is in the default state, the endowment is $y^D(s) \leq y(s)$, where a strict inequality represents the output lost after default. Specifically, let

$$V^D(s) = u\left(y^D(s)\right) + \beta \mathbb{E}V^D(s'),$$

Note that $V^D(s) \leq V^{NA}(s, 0)$ for all $s \in \mathbb{S}$.

Now suppose that default is punished by the same lost endowment, but with constant hazard θ , the government's liabilities are forgiven and it regains access to bond markets.¹² Let \tilde{V}^D denote the associated default value conditional on an equilibrium repayment value function V^R :

$$\tilde{V}^D(s) = u\left(y^D(s)\right) + \beta(1 - \theta)\mathbb{E}\tilde{V}^D(s') + \theta\beta\mathbb{E}V^R(s', 0).$$

Let us define by v_0 the expected gain from reentry:

$$v_0 \equiv \mathbb{E}\left[V^R(s, 0) - V^D(s)\right] \geq 0.$$

Manipulating the expressions for V^D and \tilde{V}^D , we find that

$$\tilde{V}^D(s) = V^D(s) + \gamma v_0, \quad \text{where } \gamma \equiv \frac{\theta\beta}{1 - \beta(1 - \theta)}.$$

In order to show uniqueness, we proceed as follows. As a first step, we take v_0 as a primitive of the environment and show that for a given v_0 , there is a unique equilibrium of the model. This step follows the same arguments as in the previous analysis. This implies a mapping from v_0 to an equilibrium value function. Consistency requires that $v_0 = \mathbb{E}[V^R(s, 0|v_0) - V^D(s)]$, where $V^R(s, b|v_0)$ is the equilibrium value of repayment conditional on the posited v_0 . The final step is to show there is a unique v_0 that satisfies this equation.

Given a value of v_0 , we can write the problem of the government as follows:

$$V^R(s, b|v_0) = \max_{c \in [0, \bar{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s') \max \left\{ V^R(s', b'), V^D(s') + \gamma v_0 \right\} \right\}$$

subject to:

$$c \leq y(s) - b + b'R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s') \mathbb{1}_{\{V^R(s', b') \geq V^D(s') + \gamma v_0\}} \right],$$

$$b' \leq \bar{B},$$

¹²The initial quantitative work of [Aguilar and Gopinath \(2006\)](#) and [Arellano \(2008\)](#) both assume such a stochastic reentry process.

where, as in the benchmark (G'), we have substituted prices using the break-even condition.

Conditional on v_0 , this problem is isomorphic to the benchmark (G'); the only difference is that V^D is translated by a constant γv_0 . It is helpful to define $\tilde{V}^R(s, b|v_0) \equiv V^R(s, b|v_0) - \gamma v_0$. Using the above, we can write that

$$\tilde{V}^R(s, b|v_0) = \max_{c \in [0, \bar{c}], b'} \left\{ u(c) - (1 - \beta)\gamma v_0 + \beta \sum_{s' \in \mathbb{S}} \pi(s') \max \{ \tilde{V}^R(s', b'|v_0), V^D(s') \} \right\}$$

subject to:

$$c \leq y(s) - b + b'R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s') \mathbb{1}_{\{\tilde{V}^R(s', b'|v_0) \geq V^D(s')\}} \right],$$

$$b' \leq \bar{B},$$

In this translated notation, the consistency condition is $(1 - \gamma)v_0 = \mathbb{E} [\tilde{V}^R(s', 0|v_0) - V^D(s')]$. The payoff of the translated problem is that $\tilde{V}^R(s, b|v_0)$ is decreasing in v_0 , a feature we now prove.

As in our analysis before, $\tilde{V}^R(s, b|v_0)$ is strictly decreasing in b for $-(\bar{A}(s), \bar{b}(s)]$, where $\bar{A}(s)$ is as defined before, and $\bar{b}(s)$ is such that $\tilde{V}^R(s, \bar{b}(s)|v_0) = u(c) + \beta\bar{V} - \gamma v_0$.

We exploit the dual representation to show that \tilde{V}^R is decreasing in v_0 . Let $\tilde{B}(s, v|v_0)$ be the inverse of $\tilde{V}^R(s, b|v_0)$ on the translated domain $\tilde{\mathbb{V}} \equiv [u(c) + \beta\bar{V} - \gamma v_0, \bar{V} - \gamma v_0]$. Assumption 3 still implies that the continuation value of $u(c) + \beta\bar{V} - \gamma v_0$ triggers default, as $v_0 \geq 0$. Thus, all of our conditions from the previous analysis apply, and \tilde{B} is a fixed point of the following operator:

$$(Tf|v_0)(s, v) = \max_{c \in [0, \bar{c}], b', \{w(s')\}_{s' \in \mathbb{S}}} \left\{ y(s) - c + R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s') \mathbb{1}_{\{w(s') \geq V^D(s')\}} \right] b' \right\}$$

subject to:

$$v \leq u(c) - (1 - \beta)\gamma v_0 + \beta \sum_{s' \in \mathbb{S}} \pi(s') \max \{ w(s'), V^D(s') \}$$

$$b' \leq f(s', w(s')) \text{ for all } s' \in \mathbb{S} \text{ such that } w(s') \geq V^D(s')$$

$$w(s') \in \tilde{\mathbb{V}} \text{ for all } s' \in \mathbb{S}.$$

As in the benchmark environment, this operator is a contraction, given v_0 . Hence, it provides a mapping from v_0 to a set of *unique* values, $x(s|v_0) = \tilde{V}^R(s, 0|v_0)$ for all $s \in \mathbb{S}$. If v_0 satisfies $\mathbb{E} [x(s|v_0) - V^D(s)] = (1 - \gamma)v_0$, then we have an equilibrium. The question is whether there are multiple values of v_0 that satisfy this consistency condition. To answer this, we first note that $\tilde{B}(s, v|v_0)$ is monotonic in v_0 :

Lemma 10. $\tilde{B}(s, v|v_0)$ is decreasing in v_0 .

Proof. Consider two values of v_0 : a, b , where $a < b$, and let B_a and B_b be the corresponding fixed points of $T(\cdot|a)$ and $T(\cdot|b)$. Then

$$\tilde{B}_a = T(\tilde{B}_a|a) \geq T(\tilde{B}_a|b).$$

Given that $T(\cdot|b)$ is a monotone operator (and a contraction), iterating on the above expression implies that

$$\tilde{B}_a \geq \lim_{n \rightarrow \infty} T^n(\tilde{B}_a|b) = \tilde{B}_b.$$

□

Recall that equilibrium consistency requires that

$$(1 - \gamma)v_0 = \mathbb{E} [x_0(s|v_0) - V^D(s)], \quad (15)$$

where $x_0(s|v_0)$ are values such that $\tilde{B}(s, x_0(s|v_0)|v_0) = 0$ for all $s \in S$. The monotonicity of \tilde{B} with respect to v and v_0 implies that, as v_0 increases, $x_0(s|v_0)$ must decrease to maintain $\tilde{B}(s, x_0(s|v_0)|v_0) = 0$. Hence, the right-hand side of equation (15) is decreasing in v_0 . The left-hand side is, however, strictly increasing in v_0 . Hence, there is a unique v_0 that is consistent with equation (15). Thus, there is a unique Markov perfect equilibrium in the model with *iid* reentry.

6 Conclusion

We have shown that a dual approach to characterizing the Markov-perfect equilibria of the Eaton-Gersovitz incomplete markets sovereign debt model implies that the inverse of the equilibrium value function is a fixed point of a contraction mapping. This result implies the uniqueness of equilibrium in the Eaton-Gersovitz model, can be used to show its existence, and may potentially be useful in numerical analysis.

The fact that the operator resembles an optimal contracting problem between lenders and the government, subject to an additional implementability condition capturing the market incompleteness, sheds light on the efficiency properties of the model's unique equilibrium.

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A Proofs

A.1 Proof of Lemma 1

We prove each part of the lemma:

Part (i). Let $C^{NA}(s, b)$ and $B^{NA}(s, b)$ denote the optimal consumption and debt policies of problem (NA) for $b \leq 0$, which exist by standard arguments. Such a policy is feasible in an equilibrium for any $b \leq 0$, as $B^{NA}(s, b) \leq 0$ and the corresponding equilibrium price is R^{-1} . It follows that, for all $b \leq 0$,

$$V^R(s, b) \geq u(C^{NA}(s, b)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V^R(s', B^{NA}(s, b))$$

Iterating this equation forward, using that $B^{NA}(s, b) \leq 0$, we obtain that $V^R(s, b) \geq V^{NA}(s, b)$ for all $b \leq 0$.

Assumption 4 then implies that $V^R(s, b) \geq V^{NA}(s, b) \geq V^D(s)$ for all $s \in \mathbb{S}$ and $b \leq 0$.

Part (ii). For $b \leq 0$, the result is immediate from $V^R(s, 0) \geq V^D(s)$ for all $s \in S$. For $b > 0$, suppose that this is not the case, and $V^R(s', b') < V^D(s')$ for all s' . This implies that default is occurring with probability one next period. As a result, the price of the bonds is $q(s, b') = 0$. Thus, from the budget constraint, we have that

$$c \leq y(s) - b.$$

Now consider the alternative policy of issuing zero bonds, $\hat{b}' = 0$. That policy can attain the same consumption level (as it generates the same budget constraint), and the value under this

alternative policy is

$$\begin{aligned}
& u(c) + \sum_{s' \in \mathbb{S}} \pi(s'|s) V^R(s', 0) \\
& \geq u(c) + \sum_{s' \in \mathbb{S}} \pi(s'|s) V^D(s') \\
& = u(c) + \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \{V^R(s', b'), V^D(s')\} \\
& = V^R(s, b).
\end{aligned}$$

where the second line follows from Part (i) and the third from the premise that $V^R(s', b') < V^D(s')$ for all $s' \in \mathbb{S}$. Hence, such $\hat{b}' = 0$ is a strict improvement (a contradiction) or also constitutes an optimal policy.

A.2 Proof of Lemma 2

We proceed to prove in each statement individually.

If $b \leq -\bar{A}(s)$, then $V^R(s, b) = \bar{V}$. Start from state (s, b) , with $b \leq -\bar{A}(s)$, and consider the strategy of setting $c = \bar{c}$ and $b' = \frac{R}{R-1}(\bar{c} - \underline{y}) = -\max_{s \in \mathbb{S}} \bar{A}(s) < 0$. As $b' < 0$, $q(s, b') = R^{-1}$ and the budget constraint is satisfied:

$$\begin{aligned}
& y(s) - b + R^{-1}b' \\
& \geq y(s) + \bar{A}(s) - (\bar{c} - \underline{y})/(R - 1) \\
& = \bar{c},
\end{aligned}$$

where the last equality uses the definition of $\bar{A}(s)$. Hence, $c = \bar{c}$ is feasible. As $b' \leq -\bar{A}(s')$ for all $s' \in \mathbb{S}$, the same policy is feasible the following period. It then follows that consuming \bar{c} indefinitely is feasible and achieves the highest possible utility level, \bar{V} .

If $b > -\bar{A}(s)$, then $V^R(s, b) < \bar{V}$. Suppose, to generate a contradiction, that $V^R(s, b) = \bar{V}$. To achieve this value, consumption must equal \bar{c} , *independently of the sequence of realized shocks in future periods*. Consider the sequence with $y = \underline{y}$ for the next k periods. Iterating on the budget set with $c = \bar{c}$ implies there exists a $k < \infty$ such that debt exceeds \bar{B} , violating the no-Ponzi condition.

If $c = \bar{c}$ for any state $(s, b) \in \mathbb{X}$, then $b < 0$. If $c = \bar{c}$ is feasible, then there exists a $b'_1 \leq \bar{B}$ such that

$$\begin{aligned}\bar{c} &\leq -b + y(s) + q(s, b'_1)b'_1 \\ &\leq -b + y(s) + \sup_{b' \leq \bar{B}} q(s, b')b' \\ &= -b + B^F(s),\end{aligned}$$

where the last equality uses Definition 1. Thus, $b \leq B^F(s) - \bar{c} < 0$, where the last inequality uses Assumption 2.

A.3 Proof of Lemma 3

We proceed to prove each part.

Part (i). Note that the constraint set in (G) is shrinking in b for $b < B^F(s)$, where $B^F(s)$ is the maximal debt level that is feasible to repay. It then follows that, for any s , $V^R(s, b)$ is weakly decreasing in b for $b < B^F(s)$.

Parts (ii) and (iii). As $V^R(s, -\bar{A}(s)) = \bar{V} > u(\underline{c}) + \beta\bar{V} > \lim_{b \downarrow B^F(s)} V^R(s, b)$, there exist thresholds $\bar{b}(s) < B^F(s)$ such that $V^R(s, b) > u(\underline{c}) + \beta\bar{V}$ for $b < \bar{b}(s)$ and $V^R(s, b) < u(\underline{c}) + \beta\bar{V}$ for $b > \bar{b}(s)$.

To establish continuity, consider a point $b_0 \leq \bar{b}(s)$. Let $b_1 = b_0 - \epsilon$ for ϵ such that $\underline{c}/2 > \epsilon > 0$. Let (c_1, b'_1) be an optimal policy for state (s, b_1) . As $b_1 < b_0 \leq \bar{b}(s)$, we have $u(\underline{c}) + \beta\bar{V} < V^R(s, \bar{b}(s)) \leq V^R(s, b_1)$. This, combined with $V^R(s, b'_1) \leq \bar{V}$, requires $c_1 > \underline{c}$.

Now consider $b_2 = b_0 + \epsilon$. Consider the debt choice $b' = b'_1$ starting from b_2 . The associated consumption is $c_1 + b_1 - b_2 = c_1 - 2\epsilon > c_1 - \underline{c} > 0$. Hence, this is feasible. This implies $V^R(s, b_1) + u(c_1 - 2\epsilon) - u(c_1) \leq V^R(s, b_2) \leq V^R(s, b_1)$, where the last inequality follows from weak monotonicity. As u is a continuous function and $c_1 - 2\epsilon$ is bounded away from zero, $V^R(s, b_2) \rightarrow V^R(s, b_1)$ as $\epsilon \rightarrow 0$. Thus, $V^R(s, b)$ is continuous for all $b_0 \leq \bar{b}(s)$ and part (iii) is proved. The fact that $V^R(s, \bar{b}(s)) = u(\underline{c}) + \beta\bar{V}$, which is part (ii) of the lemma, follows directly from continuity.

A.4 Proof of Lemma 4

The proof is by contradiction. In particular, in contradiction to the lemma, consider the following premise: for some $s \in S$, there exist b_0, b_1 , with $b_0 < b_1 \leq \bar{b}(s)$ such that $V^R(s_0, b_0) = V^R(s_0, b_1)$. We establish a number of results based on this premise:

Claim 1. *The equilibrium policy at (s, b_1) sets consumption to its upper bound: $c_1 = \bar{c}$.*

Proof. Let b'_1 denote an optimal debt choice at b_1 associated with c_1 . If $c_1 < \bar{c}$, then it is feasible at b_0 to issue b'_1 while consuming $c_0 = \min\{c_1 + b_1 - b_0, \bar{c}\} > c_1$. This yields a value strictly greater than $V^R(s, b_1)$, contradicting the premise. \square

The next claim is that the continuation value following b_1 is flat in the neighborhood below an optimal debt choice b'_1 in states of repayment:

Claim 2. *If b'_1 is an optimal debt policy at (s, b_1) , then for all $s' \in \mathbb{S}$ such that $V^R(s', b'_1) \geq V^D(s')$ and $b' \in (b'_1 - R(b_1 - b_0), b'_1)$, we have $V^R(s', b') = V^R(s', b'_1)$.*

Proof. By weak monotonicity, $V^R(s', b') \geq V^R(s', b'_1)$ for all $s' \in \mathbb{S}$ if $b' < b'_1$. Now suppose, contrary to the claim, that there is an $\hat{s} \in \mathbb{S}$ and $b' \in (b_1 - R(b_1 - b_0), b_1)$ such that $V^R(\hat{s}, b') > V^R(\hat{s}, b'_1) \geq V^D(\hat{s})$. Consider then the following policy in state (s, b_0) : $c = \bar{c}$ and $b'_0 = b'$. To see that this is feasible, recall that \bar{c} is the consumption policy for b_1 . Hence,

$$\begin{aligned} \bar{c} &\leq y(s) - b_1 + q(s, b'_1)b'_1 \\ &\leq y(s) - b_1 + q(s, b')b'_1 \\ &= y(s) - b_0 + q(s, b')b' - (b_1 - b_0) + q(s, b')(b'_1 - b') \\ &\leq y(s) - b_0 + q(s, b')b' - (b_1 - b_0) + q(s, b')R(b_1 - b_0) \\ &\leq y(s) - b_0 + q(s, b')b', \end{aligned}$$

where the second line uses the weak monotonicity of $q(s, \cdot)$; the third line adds and subtracts b_0 and $q(s, b')b'$; the fourth line uses the fact that $b' > b'_1 - R(b_1 - b_0)$ and $q(s, b') \geq 0$; and the final line uses that $q(s, b') \leq R^{-1}$, implying $(b_1 - b_0)(q(s, b')R - 1) \leq 0$. The policy $\{\bar{c}, b'\}$ generates a value to the government that is strictly higher than $V^R(s, b_1)$:

$$\begin{aligned} &u(\bar{c}) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{V^R(s', b'), V^D(s')\} \\ &> u(\bar{c}) + \beta \pi(\hat{s}|s) V^R(\hat{s}, b'_1) + \beta \sum_{s' \neq \hat{s}} \pi(s'|s) \max\{V^R(s', b'), V^D(s')\} \\ &\geq u(\bar{c}) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{V^R(s', b'_1), V^D(s')\} \\ &= V^R(s, b_1), \end{aligned}$$

where the first strict inequality uses the premise that $V^R(\hat{s}, b') > V^R(\hat{s}, b'_1) \geq V^D(\hat{s})$; the second inequality uses $V^R(s', b') \geq V^R(s', b'_1)$ for all $s' \in \mathbb{S}$ given $b' < b'_1$, as well as $V^R(\hat{s}, b'_1) \geq V^D(\hat{s})$; and the final line uses the fact that \bar{c}, b'_1 is an optimal policy for (s, b_1) . As $\{\bar{c}, b'\}$ is feasible for b_0 , we have $V^R(s, b_0) > V^R(s, b_1)$, a contradiction of our premise. \square

This implies the following:

Claim 3. *An optimal policy for (s, b_1) involves consuming \bar{c} for all future periods.*

Proof. Suppose b'_1 is an optimal debt policy at (s, b_1) . Let $\hat{S} \equiv \{s' \in \mathbb{S} | V^R(s', b'_1) \geq V^D(s')\}$. From Lemma 1 Part (ii), we can choose a b'_1 such that \hat{S} is not empty. From the previous claim, for any $s' \in \hat{S}$, $V^R(s', b'_1)$ is flat in the neighborhood below b'_1 . Hence, there exists a $b' < b'_1$ such that $V^R(s', b') = V^R(s', b'_1)$. This replicates the initial scenario, and hence $c = \bar{c}$ in state (s', b'_1) for $s' \in \hat{S}$. From Lemma 2, this implies $b'_1 < 0$. Lemma 1 Part (i) states that $V^R(s', b'_1) \geq V^D(s')$ for all $s' \in \mathbb{S}$, hence $\hat{S} = \mathbb{S}$. Thus, for all $s' \in \mathbb{S}$, we can repeat the above arguments to establish that at (s', b'_1) the government consumes \bar{c} and issues b''_1 such that $V^R(s'', b''_1)$ is flat in any state s'' following s' . Iterating forward, \bar{c} is the optimal consumption plan for all future periods following (s, b_1) . \square

Collecting results, under the premise, consumption is \bar{c} for all periods following initial state (s, b_1) . However, by Lemma 2, this requires $b_1 \leq -A(s)$, which generates a contradiction to the lemma's "if" statement. Hence, for all $b > -\bar{A}(s)$, the function $V^R(s, b)$ is strictly decreasing.

A.5 Proof of Lemma 5

Consider a $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$.

If $v_0 = \bar{V}$, then to satisfy constraint (4) from problem (B), it is necessary to set $c = \bar{c}$ and $V^R(s', b') = \bar{V}$ for all $s' \in \mathbb{S}$. This requires that $b' \leq -\bar{A}(s') < 0$ for all s' . Given that $b' < 0$, it is then optimal, to set $b' = \min_{s' \in \mathbb{S}} \{-\bar{A}(s')\} = (\bar{c} - \underline{y})R/(R - 1)$, where the last equality follows from (3). The objective is then

$$\hat{B}(s_0, \bar{V}) = y(s_0) - \bar{c} + \frac{1}{R-1}(\bar{c} - \underline{y}) = -\bar{A}(s_0) = B(s_0, \bar{V}),$$

where the last equality follows from Lemma 2.

Now consider $v_0 < \bar{V}$. From Lemmas 2 and 4, there exists a unique $B(s_0, v_0) = b_0 < -\bar{A}(s_0)$ such that $V^R(s_0, b_0) = v_0$.

First, we show that $B(s_0, v_0) \leq \hat{B}(s_0, v_0)$. Let c_0 and b'_0 be an associated optimal policy to problem (G'). Note that the policy (c_0, b'_0) satisfies (4) of problem (B), as it delivers the value v_0 . The budget constraint of problem (G') implies that

$$B(s_0, v_0) = b_0 \leq y(s_0) - c_0 + R^{-1} \left[\mathbb{1}_{\{b'_0 \leq 0\}} + \mathbb{1}_{\{b'_0 > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', b'_0) \geq V^D(s')\}} \right] b'_0 \leq \hat{B}(s_0, v_0),$$

where the last inequality follows from the fact that (c_0, b'_0) is feasible in problem (B).

Second, we show that $B(s_0, v_0) = \hat{B}(s_0, v_0)$. To show this, consider a situation in which $B(s_0, v_0) < \hat{B}(s_0, v_0)$. Then, there exists (\hat{c}_0, \hat{b}'_0) , a policy in problem (B) that delivers some objective $\hat{b}_0 > B(s_0, v_0) = b_0$. Rearranging the objective in (B) evaluated at the policy, we have

$$\begin{aligned} \hat{c}_0 &= y(s_0) - \hat{b}_0 + R^{-1} \left[\mathbb{1}_{\{\hat{b}'_0 \leq 0\}} + \mathbb{1}_{\{\hat{b}'_0 > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', \hat{b}'_0) \geq V^D(s')\}} \right] \hat{b}'_0 \\ &< y(s_0) - b_0 + R^{-1} \left[\mathbb{1}_{\{\hat{b}'_0 \leq 0\}} + \mathbb{1}_{\{\hat{b}'_0 > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', \hat{b}'_0) \geq V^D(s')\}} \right] \hat{b}'_0, \end{aligned} \quad (16)$$

where the second line follows from $\hat{b}_0 = \hat{B}(s_0, v_0) > b_0$. Note that (16) implies that the budget constraint of problem (G') holds for state (s_0, \hat{b}_0) : $\{\hat{c}_0, \hat{b}'_0\}$ is feasible and delivers value v_0 . Hence, $V^R(s_0, \hat{b}_0) \geq v_0 = V^R(s_0, b_0)$. By monotonicity of V^R , $\hat{b}_0 \leq b_0$, a contradiction.

A.6 Proof of Lemma 6

Let V^R be an equilibrium value function with inverse B . We need to show that $(TB)(s, v) = B(s, v)$ for all $(s, v) \in \mathbb{S} \times \mathbb{V}$.

First we show that $TB \geq B$. Consider a state $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$. Note that the constraint set of problem (B) is non-empty (for example, set $b' = \min_{s'} \{-\bar{A}(s')\} \leq \bar{B}$ and let $c \in [\underline{c}, \bar{c}]$ be such that (4) is satisfied).

Let (c_0, b'_0) be an element of the constraint set in problem (B) given that state. For each $s' \in \mathbb{S}$ such that $V^R(s', b'_0) \geq V^D(s')$, define $w_0(s') \equiv V^R(s', b'_0)$.

This implies $w(s') \in [V^D(s'), \bar{V}] \subset \mathbb{V}$ by Assumption 3. For all other s' such that $V^R(s', b'_0) < V^D(s')$, we let $w_0(s')$ be arbitrary elements of $(u(\underline{c}) + \beta \bar{V}, V^D(s'))$.

We now argue that the choice $(c_0, b'_0, \{w_0(s')\})$ satisfies the constraint set of problem (T) when $f = B$ given state (s_0, v_0) .

For constraint (7), note that

$$\begin{aligned} b'_0 &= B(s', w_0(s')) \text{ if } b'_0 \geq -\bar{A}(s') \\ b'_0 &\leq B(s', w_0(s')) \text{ if } b'_0 < -\bar{A}(s') \end{aligned}$$

for all s' such that $w(s') \geq V^D(s')$; hence, constraint (7) is satisfied. Note that $(c_0, b'_0, \{w_0(s')\})$ satisfies constraint (6) with equality.

Hence, for any $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$, and for any feasible choice, (c_0, b'_0) in problem (B), there exists a policy, $(c_0, b'_0, \{w_0(s')\})$ that is feasible in problem (T) given the state (s_0, v_0) when $f = B$ and attains the same value for the objective. It follows that $(TB)(s, v) \geq B(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$.

Next, we show that $TB \leq B$. Given a state $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$, consider a feasible choice $(c_0, b'_0, \{w_0(s')\})$ of problem (T) (that is, it satisfies the constraints of that problem) when $f = B$. Let us consider a policy (\hat{c}, \hat{b}') for problem (B). We now check that we can construct a feasible policy where $\hat{b}' = b'_0$ and $\hat{c} \geq c_0$.

- (i) Constraint (5) holds with b'_0 : $b'_0 \leq \bar{B}$.

Note that for any s' such that $w_0(s') \geq V^D(s')$, constraint (7) implies $b'_0 \leq B(s', w_0(s')) \leq \bar{B}$. If $w_0(s') < V^D(s')$ for all $s' \in \mathbb{S}$, constraint (7) is not relevant and any $b'_0 \geq 0$ delivers the same objective; hence, we can consider an arbitrary $b'_0 \leq \bar{B}$.

- (ii) There exists a $\hat{c} \in [\underline{c}, c_0]$ such that (\hat{c}, b'_0) satisfies (4) with equality.

First, consider $s' \in \mathbb{S}$ such that $w(s') \geq V^D(s')$. For all such states, constraint (7) evaluated at $f = B$ implies $\bar{b}(s') \geq B(s', w(s')) \geq b'_0$. Now, $V^R(s', B(s', w(s'))) \leq V^R(s', b'_0)$, as V^R is

monotonic. Using that B is the inverse of V^R , it follows that $w(s') \leq V^R(s', b'_0)$ for $s' \in \mathbb{S}$ such that $w(s') \geq V^D(s')$. This implies

$$\begin{aligned}
& u(c_0) + \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{V^R(s', b'_0), V^D(s')\} \\
& \geq u(c_0) + \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{w(s'), V^D(s')\} \\
& \geq v_0 \\
& \geq u(\underline{c}) + \beta \bar{V} \\
& \geq u(\underline{c}) + \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{V^R(s', b'_0), V^D(s')\}
\end{aligned}$$

where the third line follows from (6) and the fourth line from $v_0 \in \mathbb{V}$. Hence, by continuity of u , there exists $\hat{c} \in [\underline{c}, c_0]$ such that

$$v_0 = u(\hat{c}) + \sum_{s' \in \mathbb{S}} \pi(s'|s) \max\{V^R(s', b'_0), V^D(s')\}$$

A similar argument guarantees the existence of such $\hat{c} \in [\underline{c}, c_0]$ when $w(s') < V^D(s')$ for all $s' \in \mathbb{S}$.

Hence, (\hat{c}, b'_0) satisfies the constraints in problem (B) for (s_0, v_0) . From the optimization in problem (B),

$$\begin{aligned}
B(s_0, v_0) & \geq y(s_0) - \hat{c} + R^{-1} \left[\mathbb{1}_{\{b'_0 \leq 0\}} + \mathbb{1}_{\{b'_0 > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b'_0) \geq V^D(s')\}} \right] b'_0 \\
& \geq y(s_0) - c_0 + R^{-1} \left[\mathbb{1}_{\{b'_0 \leq 0\}} + \mathbb{1}_{\{b'_0 > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{w(s') \geq V^D(s')\}} \right] b'_0.
\end{aligned}$$

The final line is the objective in (T). As $(c_0, b'_0, \{w_0(s')\})$ was arbitrary, we have $B(s_0, v_0) \geq (TB)(s_0, v_0)$.