

Self-Fulfilling Debt Dilution: Maturity and Multiplicity in Debt Models*

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Abstract

We establish that creditor beliefs regarding future borrowing can be self-fulfilling, leading to multiple equilibria with markedly different debt accumulation patterns. We characterize such indeterminacy in the Eaton-Gersovitz sovereign debt model augmented with long maturity bonds. Two necessary conditions for the multiplicity are: (i) the government is more impatient than foreign creditors, and (ii) there are deadweight losses from default. The multiplicity is dynamic and stems from the self-fulfilling beliefs of how future creditors will price bonds; long maturity bonds are therefore a crucial component of the multiplicity. We introduce a third party with deep pockets to discuss the policy implications of this source of multiplicity and identify the potentially perverse consequences of traditional “lender of last resort” policies.

1 Introduction

The recent sovereign debt crisis in Europe, along with the associated policy responses, underscores the importance of self-fulfilling debt crises. We introduce and analytically solve a tractable version of the canonical Eaton and Gersovitz (1981) sovereign debt model with long duration

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bonds and study the vulnerability to self-fulfilling debt crises. The Eaton-Gersovitz model, enhanced to incorporate long-term bonds, has become the workhorse paradigm for a large quantitative literature that has successfully explained key empirical features of sovereign default.¹ However, due to the intractability of the model, it is not known whether and under what circumstances this environment generates self-fulfilling debt crises.² This is a major shortcoming, as long-term bonds are the primary source of government financing around the world. Moreover, they play a key role in bringing the quantitative sovereign debt models closer to the data, in large part due to the inherent incentive to dilute bondholders. We establish that the same force generates multiplicity.

Our analysis introduces a tractable version of the Eaton-Gersovitz model for which we solve for equilibrium objects explicitly. We show that as long as the government is relatively impatient and there are deadweight costs to default, there is a parameter configuration and a maturity of debt that supports multiple equilibria.

The multiplicity is dynamic. Creditor expectations of future borrowing and default behavior determine bond prices today. In turn, current and anticipated bond prices affect the government's incentives to borrow. To shed light on this feedback mechanism, we characterize two types of equilibria with markedly different debt dynamics. In a "borrowing" equilibrium, the government issues bonds until it reaches an endogenous debt limit. In a "saving" equilibrium, the government reduces its stock of debt until default no longer occurs with positive probability. The tension at work in both equilibria is the relative impatience of the government and the deadweight costs of default.

The government saves in order to enjoy high prices when it rolls over the remaining debt in the future. However, this incentive is only operable if there is a deadweight loss in default; as prices are actuarially fair in any equilibrium, they do not provide an incentive to save when default is zero sum.³ Hence, the combination of deadweight costs and the need to roll over maturing debt provides the foundation for the saving equilibrium.

The government's relative impatience provides a countervailing force that supports the borrowing equilibrium. In the borrowing equilibrium, creditors anticipate future borrowing going forward (that is, "debt dilution"), and prices are low regardless of the current level of indebted-

¹Examples, among many others, are Aguiar and Gopinath (2006), Arellano (2008), Yue (2010), Hatchondo and Martinez (2009), Mendoza and Yue (2012), Chatterjee and Eyigungor (2012), Arellano and Ramanarayanan (2012), and Bianchi, Hatchondo and Martinez (forthcoming). See Aguiar and Amador (2014) for a survey.

²In a recent contribution, Auclert and Rognlie (2016) show that the Eaton-Gersovitz model *with one-period bonds* features a unique equilibrium, but their arguments do not extend to long-term bonds.

³While lenders receive zero in the default state, a deadweight cost implies the government's value is strictly less than that associated with zero debt. Competitive bond markets imply that creditors are compensated in expectation for the full loss, while the government does not reap the same expected gain. This provides the government with an incentive to reduce the probability of default.

ness. In this equilibrium, there is no reward for keeping debt low due to creditor beliefs about future debt dynamics. Hence, whether relative impatience or deadweight costs of default are the dominant force in determining debt dynamics depends on creditor beliefs.

Maturity plays a key role in this indeterminacy, which arises only when debt is of intermediate maturity. When maturity is sufficiently long, the saving equilibrium cannot be supported, as the amount of debt to be rolled over at high prices is too small to warrant saving. In particular, as the probability of default is reduced, the gain from the reduction in the deadweight costs of default is split between the government, which is issuing new debt at high prices, and holders of non-maturing bonds, who enjoy a capital gain. As the latter component is irrelevant for the government's decision to save, longer maturity bonds eliminate the government's incentive to save.

Conversely, at very short maturities, the government internalizes the gains from reducing the probability of default. In fact, we show that as maturity becomes arbitrarily short, the government's fiscal policy approaches what would be chosen in a constrained efficient contract between the lenders and the government, as in Aguiar, Amador, Hopenhayn and Werning (2018). In this case, the borrowing equilibrium becomes impossible to sustain without a high degree of relative impatience or zero deadweight costs. For intermediate values of maturity, impatience, and deadweight costs, *either equilibrium can be sustained*.

We show that this multiplicity has novel implications for the design of third-party programs to eliminate inefficient equilibria. Common prescriptions motivated by rollover crisis intuition, such as price floors or emergency lending when spreads are high, may have the perverse outcome of eliminating the preferred equilibrium in the Eaton-Gersovitz model. In our framework, a floor on prices does not eliminate the borrowing equilibrium; in fact, it may eliminate the saving equilibrium and select the borrowing equilibrium. The saving equilibrium requires a steep gradient in prices across the domain of debt to incentivize saving (or prevent dilution). A price floor that extends across a wide range of debt levels eliminates this important feature of the saving equilibrium. A more effective policy to prevent borrowing would be to either limit debt explicitly or promise a price floor conditional on remaining within an exogenous bound on debt that is strictly tighter than the equilibrium debt limit. Such a policy would select the saving equilibrium and not require on-equilibrium resources. However, as with the lender of last resort, off-equilibrium credibility is key. The failure of such explicit debt limits in Europe (and traditional conditionality of the IMF) suggests that such credibility is difficult to establish in practice.⁴

The recent literature exploring multiplicity has built on two canonical frameworks, namely, the works of Calvo (1988) and Cole and Kehoe (2000). The Calvo multiplicity arises due to the

⁴Bocola and Dovis (2016) explore the efficacy of a price floor in a quantitative model of the European debt crisis. The policy they consider to rule out rollover crises similarly imposes a price floor combined with a debt limit.

feedback of prices to the budget set. This is easiest to see in a framework in which the government is forced to raise a certain amount of revenue from a bond auction. A low price (or high spread) for bonds forces the government to issue a greater quantity of debt in terms of face value. This raises the debt payments going forward, increasing the incentive to default and therefore supporting the low price at auction. Conversely, a high price requires lower debt payments and thus may also be an equilibrium. Calvo-style multiplicity is studied in dynamic settings by Lorenzoni and Werning (2013) and Ayres, Navarro, Nicolini and Teles (2015).

Lorenzoni and Werning (2013) provide an antecedent to our paper by analyzing the role of long-term bonds in dynamic settings with multiple equilibria. In an environment where the government follows a fiscal rule, they show how Calvo-style multiplicity arises and how longer debt maturity contributes towards uniqueness.⁵ Closer to our current environment, they also discuss how multiplicity arises in a model where the government endogenously chooses its expenditures (rather than following a pre-specified rule), but faces constraints in its ability to reduce the deficit when confronted by adverse bond prices. They uncover an equilibrium where the government saves and bond prices are high, as well as other equilibria where the government instead borrows and prices are low (when the debt is high enough). However, as we discuss in detail in Section 6, the mechanism we identify as generating multiplicity is distinct. Lorenzoni and Werning (2013) emphasizes the *limits* to fiscal discretion when bond prices are low. In contrast, we emphasize the *absence* of such limits. The fact that limited commitment to fiscal policy gives rise to multiplicity when the government issues bonds of intermediate maturities is the novel insight of our paper.

The Cole-Kehoe multiplicity is a “static” multiplicity. Specifically, holding future equilibrium behavior constant, the market clearing price for bonds is not determined. A high price for bonds allows the government to roll over its maturing debt. However, a zero price forces the government to repay all maturing bonds out of current endowment, making default optimal.⁶ This type of multiplicity has been extended recently by Aguiar, Chatterjee, Cole and Stangebye (2017) and explored quantitatively by Bocola and Dovis (2016). In our framework, the multiplicity is inherently dynamic in that future expectations over future equilibrium behavior are crucial in supporting the alternative equilibria. The Cole-Kehoe multiplicity emphasizes the vulnerability of short-maturity bonds to crises and favors lengthening maturity to avoid self-fulfilling crises. Our analysis shows that such lengthening opens up the economy to both inefficiencies and a new form of multiplicity.

⁵In their benchmark model, they show that issuing longer maturities shrinks the region of multiplicity and helps select the “good” equilibrium, while the reverse is true in our analysis.

⁶A related point on the possibility of a liquidity crisis in sovereign debt markets had been made by Sachs (1984) in a model with bank lending. Defaulting because of the inability to roll-over maturing debt generates coordination failures on the lenders side. Detragiache (1996) presents a related analysis of how investment can also generate multiple equilibria. In both of these papers, the multiplicity arises even with finite horizons.

A recent pair of papers, Stangebye (2015) and Stangebye (2018), shares our interest in multiplicity in a Eaton-Gersovitz framework. Stangebye computationally constructs a version in which there is a unique Markov equilibrium. However, he shows that introducing a sunspot may lead to self-fulfilling “panics.” Our discussion is restricted to Markovian equilibria. Moreover, our analysis highlights how beliefs determine the dominant factor among the competing forces of incentives to dilute versus the potential efficiency properties of rolling over maturing bonds. Nevertheless, given the common structure, there are many points of overlap in the nature of the multiplicity studied in the two independent papers, and we view our analysis as complementary to Stangebye’s.

The rest of the paper is as follows: Section 2 lays out our benchmark analytical model; Section 3 discusses efficient allocations from a benchmark planning problem; Section 4 contains the main analysis of the alternative equilibria; Section 5 discusses the role of maturity in generating multiplicity; Section 7 explores how commonly proposed third-party policies may or may not select a particular equilibrium; and Section 8 concludes.

2 Environment

We study an infinite-horizon small open economy. Time is continuous and indexed by t . The economy receives a constant flow endowment y . Consumption and savings decisions for the economy are made by a government. The government has access to a non-contingent bond that it trades with atomistic, risk-neutral lenders. The lenders discount at the world risk-free interest rate $R = (1 + r)$. The small open economy assumption implies that R is invariant to the government’s borrowing or default decisions. Lenders are risk-neutral, atomistic, and have sufficient wealth as a group to hold an arbitrary quantity of bonds.

The asset space is restricted to a single type of bond. To incorporate maturity in a tractable manner, we follow Leland (1994), Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012) by considering random maturity bonds. A bond matures with a constant hazard rate δ , at which point a payment of 1 is required. More precisely, consider an interval of time Δt . The probability that an individual bond has not matured by the end of the interval is $e^{-\delta\Delta t}$. We assume that bonds mature independently. Appealing to a law of large numbers, this implies that a deterministic fraction δ of any non-degenerate portfolio of bonds matures each instant. The expected life span of a bond is $1/\delta$; hence, δ is a measure of (inverse) expected maturity. The advantage of this formulation is that all bonds that have yet to mature are identical; in particular, they all have the same expected maturity going forward regardless of when they were issued.

We normalize the coupon of a bond to be the risk-free rate r . That is, a bond pays a flow coupon r each instant through maturity. This implies that a risk-free bond has price 1 in equilibrium,

which serves as the upper bound on the price of the sovereign's bond.

If the government misses a coupon or principal payment, it is in default. As in Aguiar et al. (2018), the value of default is a random variable and captures any punishment that can be imposed by creditors, including lost endowment, as well as any utility costs (or benefits) to the government from defaulting. Changes in the value of default represent the source of risk to creditors in our analysis.

Specifically, let $N(t)$ denote a Poisson counting process with intensity λ .⁷ For a given path $N(t)$, let $V^D(t)$ be the associated time-path of the value of default. In particular, $V^D(t) = \underline{V}$ for all t except where $N(t)$ is discontinuous; and, at points of discontinuity, $V^D(t) = \bar{V} > \underline{V}$.

In words, the government almost always has the option to default and receive a payoff of $V^D(t) = \underline{V}$. With constant arrival probability λ , this default value temporarily increases to $V^D(t) = \bar{V} > \underline{V}$. The higher value represents an opportunity to default with lower consequences for punishment. If the government does not exercise this high default value option when it arrives, the default value returns to \underline{V} until the next arrival of \bar{V} .

To define preferences, let $\mathbf{c} = \{c(t)\}_{t \geq 0}$ denote a *deterministic* consumption stream that characterizes the government's consumption until default.⁸ We assume linear flow utility, $u(c) = c$. This allows an explicit characterization of the equilibrium objects while incorporating key economic forces that are robust to curvature in utility. Consumption at each point in time is restricted to lie in the interval $[\underline{C}, \bar{C}]$. Let \mathcal{C} denote the space of consumption sequences with $c(t) \in [\underline{C}, \bar{C}]$ for all t .⁹

Given a consumption sequence \mathbf{c} , we define the government's expected value. Let T denote the time at which the government defaults, if ever, at the low outside default value. The value to

⁷That is, $N(t)$ has independent increments, and in any interval $[t, t+s)$, the random increment $N(t+s) - N(t) \geq 0$ is a Poisson random variable with parameter λs .

⁸The fact that $c(t)$ is not indexed to $N(t)$, that is, the realization of V^D , anticipates the fact that without contingent bonds, consumption will be deterministic conditional on no default.

⁹Given our restriction that $c \in [\underline{C}, \bar{C}]$, it would be equivalent to define u for the entire real line but set $u(c) = \bar{C}$ for $c \geq \bar{C}$ and $u(c) = -\infty$ for $c \leq \underline{C}$.

the government of a consumption sequence \mathbf{c} , $V(t, \mathbf{c})$, is recursively defined by:

$$\begin{aligned}
V(t, \mathbf{c}) &= \sup_{T \geq t} \left\{ \left[\int_t^T e^{-\rho(s-t)} c(s) ds + e^{-\rho(T-t)} \underline{V} \right] e^{-\lambda(T-t)} + \right. \\
&\quad \left. \int_t^T \left[\int_t^s e^{-\rho(\tau-t)} c(\tau) d\tau + e^{-\rho(s-t)} \max\langle V(s, \mathbf{c}), \bar{V} \rangle \right] \lambda e^{-\lambda(s-t)} ds \right\} \\
&= \sup_{T \geq t} \left\{ \int_t^T e^{-(\rho+\lambda)(s-t)} c(s) ds + e^{-(\rho+\lambda)(T-t)} \underline{V} + \right. \\
&\quad \left. \lambda \int_t^T e^{-(\rho+\lambda)(s-t)} \max\langle V(s, \mathbf{c}), \bar{V} \rangle ds \right\}. \tag{1}
\end{aligned}$$

The first line is the value absent the arrival of the high default outside option, where the probability that T is reached before the first arrival of the high outside option is $e^{-\lambda(T-t)}$. The inner integral in the second term is the value conditional on the high outside option arriving at time $s < T$, which is then integrated over all possible $s \in [t, T)$. The second equality follows from straightforward integration. Standard methods verify that there is a unique bounded fixed point V that satisfies (1) given \mathbf{c} . From (1), we have immediately that $V(t, \mathbf{c}) \geq \underline{V}$ for all t and \mathbf{c} , as $T = t$ is always an option.

We make the following assumptions on the primitives of the environment:

Assumption 1. (i) $\rho \geq r$; (ii) $y \geq \rho \bar{V}$; (iii) $\bar{C} > y$; (iv) $\underline{C} < (\rho + \lambda) \underline{V} - \lambda \bar{V}$.

The first item ensures that the government is relatively impatient (as compared to the market interest rate) and does not accumulate infinite assets. The second states that consuming the endowment forever is weakly greater than the high default value. If \bar{V} were strictly greater than this value, the government may prefer default to holding a small amount of assets. When $y > \rho \bar{V}$, there is a deadweight loss in default; in particular, from the lenders' perspective, all debt is zeroed out once default occurs, but the government receives a value that is strictly less than full debt forgiveness. In the original Eaton and Gersovitz (1981), this difference reflected the loss of insurance. In the recent quantitative literature starting with Aguiar and Gopinath (2006) and Arellano (2008), an additional endowment cost is imposed during default. In the current environment, the gap $y - \rho \bar{V}$ makes default inefficient (in terms of joint borrower-lender surplus) and will play an important role in equilibrium debt dynamics. The third condition ensures that consuming the endowment is always feasible. The final condition guarantees that it is feasible to deliver the low default value to the government without imposing an immediate default.

Some of the assumptions above were made to obtain tractability and build on earlier work in this literature.¹⁰ However, the underlying economics are robust to the inclusion of endow-

¹⁰For continuous time formulations of sovereign debt models with Poisson shocks that trigger default see Aguiar,

ment risk, concave utility, and discrete time. We have constructed numerical examples of the multiplicity studied below in these extended environments.

3 Constrained Efficient Allocations

We first study an efficient allocation that maximizes the joint surplus between a risk-neutral lender and the government subject to the government's lack of commitment to repay. The efficient allocations provide a useful benchmark to understand the competitive equilibria studied in the next section.

Consider a Pareto planning problem that maximizes the expected payments to a risk-neutral lender conditional on delivering a value weakly greater than v to the government. As in Aguiar et al. (2018), the planning problem chooses a consumption stream \mathbf{c} , but the planner cannot prevent the government from defaulting when the government finds it optimal to do so. In particular, for consumption sequence \mathbf{c} , the government's value is defined by (1). When the government is indifferent to default or continuing, the planner can break the tie.

Given an allocation \mathbf{c} and time T that maximizes (1) at time $t = 0$, the expected payments to the lender can be defined as:

$$P(\mathbf{c}, T) = \int_0^T e^{-\int_0^t r + \mathbb{1}_{[V(s, \mathbf{c}) < \bar{v}]} \lambda ds} [y - c(t)] dt, \quad (2)$$

where $\mathbb{1}_{[x]}$ is an indicator function that takes value one if x is true and zero otherwise. The integrand represents the flow payments to the lender, which are discounted by r and the probability of default prior to period T . Here we have incorporated that the government does not default when indifferent upon the arrival of the high default value, which is without loss given that we will focus on Pareto efficient allocations.

Definition 1. *An allocation $\{\mathbf{c}, T\}$ is efficient if T maximizes (1) at $t = 0$ given \mathbf{c} , and if there is no alternative allocation $(\tilde{\mathbf{c}}, \tilde{T})$ such that $V(0, \tilde{\mathbf{c}}) \geq V(0, \mathbf{c})$ and $P(\tilde{\mathbf{c}}, \tilde{T}) \geq P(\mathbf{c}, T)$, with one inequality strict.*

Amador, Farhi and Gopinath (2013) and Lorenzoni and Werning (2013). For Brownian motion shocks, see Nuño and Thomas (2015), Tourre (2017), and DeMarzo, He and Tourre (2018). The latter two also discuss how linear utility (in their cases without consumption bounds) facilitates finding closed form solutions.

Toward characterizing efficient allocations, we define the following planning problem:

$$P^*(v) = \sup_{c \in \mathcal{C}, T \geq 0} P(c, T) \quad (3)$$

$$\text{subject to } \begin{cases} V(0, c) = v \\ T \text{ maximizes (1) at } t = 0. \end{cases}$$

We define P^* on the domain $v \in [\underline{V}, V_{max}] \equiv \mathbb{V}$. It is infeasible to deliver $v < \underline{V}$. It is also infeasible to deliver higher value than \bar{C}/ρ , and we assume $V_{max} < \bar{C}/\rho$.¹¹ Note that if P^* is strictly decreasing, it characterizes the Pareto frontier. In what follows, we assume \underline{C} is sufficiently low to guarantee that P^* is strictly decreasing.¹²

The first result states that we can restrict attention to allocations in which default occurs only if $V^D(t) = \bar{V}$:

Lemma 1. *It is weakly optimal in problem (3) to default only if $V^D(t) = \bar{V}$. That is, in any efficient allocation, $T = \infty$.*

This lemma allows us to substitute $T = \infty$ in (3). The value function $P^*(v)$ has the following standard properties:

Lemma 2. *The solution to the planner's problem, $P^*(v)$, is bounded and Lipschitz continuous.*

To solve problem (3), we appeal to standard recursive techniques and study the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(r + \mathbb{1}_{[v < \bar{V}]} \lambda) P^*(v) = \sup_{c \in [\underline{C}, \bar{C}]} \{y - c + P^{*\prime}(v) \dot{v}\}, \quad (\text{P})$$

subject to

$$\dot{v} = -c + \rho v - \mathbb{1}_{[v < \bar{V}]} \lambda \left[\bar{V} - v \right] \quad (4)$$

and the state-space constraint $v \in \mathbb{V}$.

¹¹The fact that V_{max} is strictly less than \bar{C}/ρ ensures that the planner can set $\dot{v} < 0$ at the upper bound of the domain, a controllability requirement used in some of our proofs.

¹²The reason why $P^*(v)$ may not be decreasing is that the threat of default is so severe that the planner would rather “forgive debt” by raising v to $v' > v(0)$ instantaneously at $t = 0$ without compensating lenders. If \underline{C} is sufficiently low, forgiveness is dominated by setting $c = \underline{C}$ until $v(t) = v'$. As $\underline{C} \rightarrow -\infty$, this approximates a lump-sum payment at $t = 0$, which allows the planner to move v arbitrarily fast relative to the first arrival of \bar{V} . Specifically, $\lim_{\underline{C} \rightarrow -\infty} (P^*(v) - P^*(v')) \geq v' - v$.

Lemma 2 states that P^* is bounded and Lipschitz continuous, and hence differentiable almost everywhere. However, there may be isolated points of non-differentiability. At such points, P^* satisfies (P) in the viscosity sense. In particular:

Proposition 1. *Suppose a bounded, Lipschitz continuous function $p(v)$ with domain \mathbb{V} has the following properties:*

(i) p satisfies (P) at all points of differentiability;

(ii) If $\lim_{v \uparrow \bar{V}} p'(v) > \lim_{v \downarrow \bar{V}} p'(v)$ and $\lim_{v \uparrow \bar{V}} p'(v) \geq -1$, then $p(\bar{V}) = (y - \rho \bar{V})/r$;

(iii) At a point of non-differentiability $\tilde{v} \neq \bar{V}$, we have $\lim_{v \uparrow \tilde{v}} p'(v) < \lim_{v \downarrow \tilde{v}} p'(v)$;

(iv) If $p'(\underline{V}) < -1$, then $p(\underline{V}) = (y - \rho \underline{V} + \lambda(\bar{V} - \underline{V}))/r + \lambda$;¹³ and

(v) $p'(V_{max}) \leq -1$;

then $p(v) = P^*(v)$.

The first condition of the proposition ensures that the candidate value function satisfies the HJB wherever it is smooth. The second condition concerns the case when \bar{V} is a locally stable stationary point; this will be relevant when we consider an efficient “saving allocation” defined below. The third condition states that any other point of non-differentiability has a “convex” kink. The final two conditions are sufficient to ensure that v remains in \mathbb{V} .

Problem (P) implies that we can divide the state space into two regions. For $v \in [\underline{V}, \bar{V})$, default occurs with probability λ . Following Cole and Kehoe (2000), we refer to this subset of the domain as the *Crisis Zone*. For $v \in [\bar{V}, V_{max}]$, default does not occur even if the high outside default value is available. We refer to this subset as the *Safe Zone*.

Let $C^*(v)$ denote an optimal policy associated with the recursive formulation. From (P), the first-order condition for an interior consumption is

$$P^{*\prime}(v) = -1. \tag{5}$$

If this condition holds, the planner is indifferent over $c \in [\underline{C}, \bar{C}]$. If $P^{*\prime}(v) < -1$, then $C^*(v) = \bar{C}$ is strictly optimal, and $C^*(v) = \underline{C}$ is optimal for $P^{*\prime}(v) > -1$.¹⁴

¹³For the endpoints of \mathbb{V} , we interpret $p'(\underline{V}) \equiv \lim_{v \downarrow \underline{V}} p'(v)$ and $p'(V_{max}) \equiv \lim_{v \uparrow V_{max}} p'(v)$.

¹⁴Note that a necessary condition for optimality in Problem (P) is that the value function is weakly higher than the value generated by a stationary policy:

$$P^*(v) \geq \frac{y - \rho v + \mathbb{1}_{[v < \bar{V}]} \lambda (\bar{V} - v)}{r + \mathbb{1}_{[v < \bar{V}]} \lambda}.$$

This is implied by the first-order condition.

To characterize Pareto efficient allocations, we proceed in steps: we conjecture a candidate efficient allocation; we solve (P) under this conjecture; and then we verify if and when the candidate allocation satisfies the optimality conditions set out in Proposition 1. Our conjectures are guided by the two competing forces driving debt dynamics; namely, relative impatience favors debt accumulation, while the costs of default favor debt reduction. The next two subsections derive solutions assuming that the borrowing and saving forces dominate, respectively. With the solutions in hand, we verify under what parameter configurations they solve the Pareto problem. We also verify that there are no parameter configurations for which neither the borrowing nor the saving solution is valid.

3.1 Efficient Borrowing Allocations

We first conjecture that the borrowing incentive dominates. Given the linearity of utility, a reasonable conjecture is that consumption is at the upper bound until v reaches \underline{V} . In particular, we define

$$C_B^*(v) \equiv \begin{cases} \bar{C} & \text{for } v \in (\underline{V}, \bar{V}] \\ (\rho + \lambda)\underline{V} - \lambda\bar{V} & \text{for } v = \underline{V}. \end{cases}$$

This sets consumption at its maximum possible level, \bar{C} , for the entire state space except at the lowest possible value \underline{V} . This implies $\dot{v} < 0$ for $v > \underline{V}$. At $v = \underline{V}$, the value cannot be further reduced given the government's option to default. Hence, consumption is set to deliver $\dot{v} = 0$. From equation (4), $\dot{v} = 0$ at \underline{V} implies that $c = (\rho + \lambda)\underline{V} - \lambda\bar{V}$.

With the conjectured consumption policy function in hand, we solve (P), using consumption at \underline{V} to pin down the boundary condition $P^*(\underline{V})$. In particular, we define

(i) For $v \in [\underline{V}, \bar{V}]$:

$$P_B^*(v) \equiv \frac{1}{r + \lambda} \left[y - \bar{C} + \frac{(\bar{C} + \lambda\bar{V} - (\rho + \lambda)v)^{\frac{r+\lambda}{\rho+\lambda}}}{(\bar{C} + \lambda\bar{V} - (\rho + \lambda)\underline{V})^{\frac{r-\rho}{\rho+\lambda}}} \right].$$

(ii) For $v \in (\bar{V}, V_{max}]$:

$$P_B^*(v) \equiv \frac{1}{r} \left[y - \bar{C} + (\bar{C} - y + rP_B^*(\bar{V})) \frac{(\bar{C} - \rho v)^{\frac{r}{\rho}}}{(\bar{C} - \rho\bar{V})^{\frac{r}{\rho}}} \right].$$

Remark 1. To gain some insight into these expressions, let us take the limit as \bar{C} becomes arbitrarily

large. In this case, for any v ,

$$\lim_{\bar{C} \rightarrow \infty} P_B^*(v) = P_B^*(\underline{V}) - (v - \underline{V}), \quad (6)$$

where $P_B^*(\underline{V}) = (y - (\rho + \lambda)\underline{V} + \lambda\bar{V})/(r + \lambda)$ for all \bar{C} . Expression (6) states that the payment to the lenders is the maximal incentive-compatible payment minus a lump sum consumed by the government in the initial period. This limiting result holds over the entire domain of v , as $\bar{C} \rightarrow \infty$, the implied dynamics become infinitely fast, and whether the initial state is in the Safe Zone or the Crisis Zone becomes irrelevant.

For P_B^* to be a solution to the planning problem requires $P_B^{*\prime}(v) \leq -1$ for $v > \underline{V}$ at points of differentiability. For $v \in [\underline{V}, \bar{V}]$, we immediately have $P_B^{*\prime}(v) \leq -1$. For $v \in (\bar{V}, V_{max}]$, note that P_B^* is strictly concave. Therefore, it suffices to check the first-order condition at $v \downarrow \bar{V}$. This turns out to be the same condition as $P_B^*(\bar{V}) \geq (y - \rho\bar{V})/r$. Summarizing:

Proposition 2. P_B^* is a solution to the planning problem if and only if

$$rP_B^*(\bar{V}) \geq y - \rho\bar{V}. \quad (7)$$

This condition has the following interpretation: it is efficient to borrow into the Crisis Zone rather than remain in the Safe Zone indefinitely. The left-hand side is the annuitized value of the objective from borrowing into the Crisis Zone. The right-hand side is the net payments to the lender from setting $\dot{v} = 0$ at the boundary of the Safe Zone. The decision of whether to exit the Safe Zone is the crucial question given the inefficiencies associated with default, and the proposition states that this is the only restriction on parameter values that needs to be checked to verify that the borrowing allocation is efficient.

To interpret this condition, consider the two forces identified above: relative impatience and the deadweight costs of default. The right-hand side of (7) represents the deadweight costs of default. The larger this is, the more costly it is to enter the Crisis Zone and the more stringent this condition.

To see the role of impatience, the left-hand side captures the value of delivering utility to the government by front-loading consumption. Moreover, when $\rho = r$, we have $P_B^*(\bar{V}) = (y - \rho\bar{V})/(r + \lambda)$. Hence, condition (7) cannot be satisfied if both $r = \rho$ and $y > \rho\bar{V}$, that is, if there is a deadweight cost to default and the government is not impatient.

Finally, note that $P_B^*(v)$ is strictly decreasing for $v \in [\underline{V}, V_{max}]$. As a result, P_B^* traces out the Pareto frontier when the borrowing allocation is efficient. It will be useful when studying the competitive equilibria to consider the inverse of P_B^* , which we denote V_B^* . The function V_B^* maps the present value of promised payments to the lender into the government's value under

the borrowing allocation. For reference:

$$V_B^*(p) = \begin{cases} \frac{1}{\rho} \left(\bar{C} - (\bar{C} - \rho \bar{V}) \left(\frac{\bar{C} - y + rp}{\bar{C} - y + r \underline{P}_B} \right)^{\frac{\rho}{r}} \right) & \text{for } p \in (-\bar{a}, \underline{P}_B] \\ \frac{1}{\rho + \lambda} \left(\bar{C} + \lambda \bar{V} - \frac{(\bar{C} - y + (r + \lambda)p)^{\frac{\rho + \lambda}{r + \lambda}}}{(\bar{C} - y + (r + \lambda) \bar{P}_B)^{\frac{\rho - r}{r + \lambda}}} \right) & \text{for } p \in (\underline{P}_B, \bar{P}_B], \end{cases} \quad (8)$$

where $\bar{a} \equiv (\bar{C} - y)/r$ is the maximal net inflows that can be consumed by the government; $\underline{P}_B \equiv P_B^*(\bar{V})$ is the threshold of the Safe Zone; and $\bar{P}_B \equiv P_B^*(\underline{V})$ is the maximal value that can be extracted from the government in a borrowing allocation.

For reference, we repeat Proposition 2 using the inverse notation:

Corollary 1. P_B^* is a solution to the planning problem if and only if

$$\bar{V} \leq V_B^* \left(\frac{y - \rho \bar{V}}{r} \right). \quad (9)$$

The left-hand side is the government's value at the Safe Zone threshold. The right-hand side is the value of the government when it promises to pay lenders $(y - \rho \bar{V})/r$ in expected value. One way to deliver this is to pay $y - \rho \bar{V}$ each period indefinitely. This implies the government's consumption is $\rho \bar{V}$, which delivers value \bar{V} , making this allocation compatible with no default. The corollary states that the planner can do weakly better than this by borrowing into the Crisis Zone.

3.2 Efficient Saving Allocations

An alternative to borrowing into the Crisis Zone is to save into the Safe Zone. This allocation favors reducing the probability of default over the relative impatience of the government.

We start then by conjecturing that the Safe Zone is an absorbing state. In particular, for the Safe Zone, we let consumption be

$$C_S^*(v) \equiv \begin{cases} \bar{C} & \text{if } v \in (\bar{V}, \bar{C}/\rho) \\ \rho \bar{V} & \text{if } v = \bar{V}. \end{cases} \quad (10)$$

This implies that in the interior of the Safe Zone, the government receives the maximal consumption. However, at the boundary, the government receives the consumption that sets $\dot{v} = 0$, and hence v never transits from the Safe Zone into the Crisis Zone.

Substituting this consumption policy into (P), and using the boundary condition $P_S^*(\bar{V}) =$

$(y - \rho\bar{V})/r$, we obtain that P_S^* is

$$P_S^*(v) \equiv \frac{1}{r} \left[y - \bar{C} + (\bar{C} - \rho\bar{V})^{\frac{\rho-r}{\rho}} (\bar{C} - \rho v)^{\frac{r}{\rho}} \right] \text{ for } v \in [\bar{V}, V_{max}]. \quad (11)$$

For the Crisis Zone, the planner decides between saving toward the Safe Zone or remaining in the Crisis Zone. We denote the former scenario with a “hat.” In particular, the linearity of the problem leads us to conjecture that if saving is efficient, consumption will be at its lower bound. Thus, we define

$$\hat{C}(v) \equiv \underline{C} \text{ for } v \in [\underline{V}, \bar{V}]. \quad (12)$$

The associated value from this policy is obtained by solving (P) using $P_S^*(\bar{V})$ as a boundary condition:

$$\hat{P}(v) \equiv \frac{1}{r + \lambda} \left[y - \underline{C} + (\underline{C} - y + (r + \lambda)P_S^*(\bar{V})) \left(\frac{\underline{C} + \lambda\bar{V} - (\rho + \lambda)v}{\underline{C} - \rho\bar{V}} \right)^{\frac{r+\lambda}{\rho+\lambda}} \right]. \quad (13)$$

Remark 2. *To gain some insight into the above expression, we take the limit as $\underline{C} \rightarrow -\infty$. In this case, the dynamics in the saving allocation become arbitrarily fast and*

$$\lim_{\underline{C} \rightarrow -\infty} \hat{P}(v) = P_S^*(\bar{V}) + \bar{V} - v. \quad (14)$$

That is, the conjectured allocation calls for an initial lump sum payment by the government that is sufficient to reach the boundary of the Safe Zone immediately.

The value from saving into the Safe Zone is one building block of the efficient saving allocation. However, the planner may find it optimal to abandon the savings strategy in the Crisis Zone and instead pursue the borrowing one. As a result, our conjectured value function in the Crisis Zone is the upper envelope of the savings and the borrowing conjectures:¹⁵

$$P_S^*(v) \equiv \max\langle \hat{P}(v), P_B^*(v) \rangle \text{ for } v \in [\underline{V}, \bar{V}]. \quad (15)$$

Using the expressions above, it is possible to show that \hat{P} and P_B^* cross at most once for $v \in [\underline{V}, \bar{V}]$. We denote by $v^I \in [\underline{V}, \bar{V}]$ such a crossing point and set $v^I = \underline{V}$ if they do not cross. The point v^I has a particular interpretation: the planner is indifferent between saving out of the Crisis Zone versus remaining in the Crisis Zone indefinitely at that point. For values of v above v^I , the

¹⁵The Cole-Kehoe model also features a savings and a borrowing region within the Crisis Zone (for certain parameter values) when the government is impatient. See, for example, Cole and Kehoe (1996), Figure 2.

planner finds it optimal to save, while for values below v^I , the planner finds it optimal to borrow. With this result in hand, we can complete the characterization of the policy function by setting

$$C_S^*(v) \equiv \begin{cases} \underline{C} & \text{if } v \in [v^I, \bar{V}) \\ C_B^*(v) & \text{if } v \in [\underline{V}, v^I). \end{cases} \quad (16)$$

We now verify under what conditions P_S^* solves the planning problem. Condition (ii) of Proposition 1 is relevant in this case, as \bar{V} is a locally stable stationary point. The crucial condition is whether at the boundary of the Safe Zone, the objective is maximized by staying put versus borrowing to the upper bound. The following is proved in Appendix B:

Proposition 3. P_S^* is a solution to the planning problem if and only if

$$rP_S^*(\bar{V}) = y - \rho\bar{V} \geq rP_B^*(\bar{V}). \quad (17)$$

Note that this condition is the mirror image of Proposition 2, which established the efficiency of the borrowing allocation. Summarizing the results of the previous two subsections, we have that efficiency is characterized by a simple condition:

Corollary 2. The borrowing allocation is efficient if and only if $P_B^*(\bar{V}) \geq (y - \rho\bar{V})/r$, and the saving allocation is efficient if and only if $P_B^*(\bar{V}) \leq (y - \rho\bar{V})/r$.

For reference, we provide the expression for the inverse of P_S^* :

$$V_S^*(p) = \begin{cases} \rho^{-1} \left(\bar{C} - (\bar{C} - \rho\bar{V}) \left(\frac{\bar{C} - y + rp}{\bar{C} - y + r\underline{P}_S} \right)^{\frac{\rho}{r}} \right) & \text{for } p \in (-\bar{a}, \underline{P}_S] \\ (\rho + \lambda)^{-1} \left(\underline{C} + \lambda\bar{V} + (\rho\bar{V} - \underline{C}) \left(\frac{y - (r+\lambda)p - \underline{C}}{\rho\bar{V} - \lambda\underline{P}_S - \underline{C}} \right)^{\frac{\rho+\lambda}{r+\lambda}} \right) & \text{for } p \in (\underline{P}_S, p^I] \\ V_B^*(p) & \text{for } p \in (p^I, \bar{P}_S], \end{cases} \quad (18)$$

where $\underline{P}_S \equiv (y - \rho\bar{V})/r = P_S^*(\bar{V})$; $\bar{P}_S \equiv P_S^*(\underline{V})$; and $p^I \equiv P_S^*(v^I)$.

4 Competitive Equilibria

We now discuss competitive equilibria, and, as we will see, the efficient allocations provide a useful benchmark in the characterization. Recall that we define V_B^* and V_S^* in equations (8) and (18) to be the inverses of P_B^* and P_S^* , respectively. Keep in mind that the domain of V^* is the expected promised value to the lender, without a notion of maturity or face value.

For the competitive equilibrium, recall that we restrict attention to non-contingent bonds with random maturity. Specifically, a bond is a promise to pay a coupon normalized to the risk-free rate r until the bond matures, which occurs with probability δdt over a vanishingly small interval of time dt .

We consider Markov equilibria. The payoff relevant states are the face value of debt b and default payoff V^D . Recall that the high default payoff state is only relevant if the government exercises the option to default; otherwise, the low default payoff state resumes. Therefore, we subsume the notation for the default payoff state $V^D = \underline{V}$ when defining prices and values conditional on repayment.

4.1 The Government's Problem

Let $V(b)$ denote the government's equilibrium value of repayment given the face value of debt b . Strategic default implies repayment if $V(b) \geq V^D$, and default otherwise.

Parallel to the analysis of Section 3, it is useful to split the state space into two regions. Given an equilibrium value V , we define the following: the *Safe Zone* is $b \in [-\bar{a}, \underline{b}]$ where \underline{b} satisfies $V(\underline{b}) = \bar{V}$ and recall that $\bar{a} \equiv (\bar{C} - y)/r$ is the upper bound on assets that can be consumed; and the *Crisis Zone* is $b \in (\underline{b}, \bar{b}]$, where \bar{b} satisfies $V(\bar{b}) = \underline{V}$. In each of the equilibria we study, we will establish the existence of the thresholds. As in the preceding analysis, the Safe Zone is the space of debt (and assets) such that the government will not default if the high default payoff state arrives. However, the government may default at some point in the future. The Crisis Zone is the space of debt such that the government will default upon the arrival of $V^D = \bar{V}$. For $b > \bar{b}$, the debt level is so high that, if the initial state is in this region, the government defaults immediately regardless of the payoff state. This region is beyond the endogenous borrowing limit and will never be reached from below in equilibrium. We denote the relevant debt state space in a competitive equilibrium by $\mathbf{B} \equiv [-\bar{a}, \bar{b}]$.

To characterize the government's problem, assume that the government faces an equilibrium price schedule $q : \mathbf{B} \rightarrow [q, 1]$, where $q > 0$ is defined below. At each point in time, the government chooses consumption as well as decides whether to pay its debt obligations or default after observing the realized V^D . Given consumption c , the government's debt evolves according to

$$q(b)[\dot{b} + \delta b] = c + (r + \delta)b - y, \quad (19)$$

where \dot{b} denotes the derivative of debt with respect to time. The left-hand side represents revenue from bond auctions, where the term in brackets is the change in the face value of debt plus the fraction of debt that matured, which is net new issuances. The terms on the right represent consumption plus payments of interest and principal minus income.

It may be the case that $q(b)$ is discontinuous at some debt level b_0 . This will occur when the government is indifferent between borrowing or saving. When indifferent, we break the tie by having the government save, which implies that the equilibrium price at b_0 is the highest of the prices consistent with the two possible strategies. For technical reasons, we place one more constraint on debt issuance policies around points of price discontinuity. We impose that for an arbitrarily small neighborhood around b_0 , debt buybacks occur at price approaching $q(b_0)$. The specifics are spelled out in Appendix B.5. Debt buybacks occur when $\dot{b} < -\delta b$, that is, when debt decreases faster than existing debt matures. Imposing that buybacks occur at the higher of the two prices around the discontinuity allows us to apply recent results in optimal control with discontinuous dynamics. Note that this condition is imposed only around points of discontinuity in the price schedule and for an *arbitrarily small* interval around them. We flag when we use this restriction in footnotes 19 and 23. In what follows, we suppress this constraint in the notation for the government's HJB equation.

We prove in Appendix Lemma A.1 that the government's value function, V , is strictly decreasing and Lipschitz continuous. In addition, it is the unique, bounded, continuous solution to the following HJB equation on \mathbf{B} , given a price schedule q :

$$(\rho + \Lambda(b))V(b) = \max_{c \in [\underline{C}, \bar{C}]} c + \underbrace{V'(b) \left(\frac{c + (r + \delta)b - y}{q(b)} - \delta b \right)}_{\dot{b}} + \Lambda(b)\bar{V}, \quad (20)$$

where

$$\Lambda(b) = \lambda \mathbb{1}_{[V(b) < \bar{V}]}. \quad (21)$$

More precisely, as in the planning problem, $V(b)$ satisfies (20) in the viscosity sense. The counterpart to Proposition 1 for the government's problem is:

Proposition 4. *Consider the government's problem given a compact debt domain \mathbf{B} and a price schedule $q : \mathbf{B} \rightarrow [q, 1]$ that has a (bounded) derivative at almost all points in \mathbf{B} . If a strictly decreasing, Lipschitz continuous function $v : \mathbf{B} \rightarrow [\underline{V}, \bar{C}/\rho]$ has the following properties:*

- (i) v satisfies (20) at all points of differentiability;
- (ii) If $\lim_{b \uparrow \underline{b}} v'(b) > \lim_{b \downarrow \underline{b}} v'(b)$, then $\rho v(\underline{b}) = \rho \bar{V} = y - [r + \delta(1 - q(\underline{b}))]\underline{b}$;
- (iii) At a point of non-differentiability $\tilde{b} \neq \underline{b}$, we have $\lim_{b \uparrow \tilde{b}} v'(b) < \lim_{b \downarrow \tilde{b}} v'(b)$;
- (iv) $\rho v(-\bar{a}) = \bar{C}$; and

$$(v) (\rho + \lambda)v(\bar{b}) = y - [r + \delta(1 - q(\bar{b}))]\bar{b} + \lambda\bar{V};$$

then $v(b) = V(b)$ is the government's value function.

The conditions listed in the proposition are similar to those from Proposition 1. Namely, that the value function satisfies the HJB equation with equality wherever smooth; there may be a local attractor that corresponds to \underline{b} if the government saves; other points of non-differentiability have convex kinks; and the endpoints of the domain deliver the value of holding debt constant.¹⁶

4.2 The Lenders' Problem

The equilibrium condition from the lenders' problem is that lenders must be indifferent to purchasing the government's bonds versus holding risk-free assets that return R . We consider $b \leq 0$ to represent risk-free assets held abroad that have a price of one. For $b > 0$, b represents the liabilities of the government. To price debt in equilibrium, consider starting from a debt level $b > 0$, and using the government's policy $C(b)$ and the budget constraint (19) to derive the equilibrium path of debt going forward, $b(t)$. The present value "break-even" bond pricing equation for the lender is

$$q(b) = \int_0^\infty e^{-(r+\delta)t - \int_0^t \Lambda(b(s))ds} (r + \delta) dt. \quad (22)$$

The integrand is the coupon payment r plus principal δ . The discount factor is the interest rate r plus the rate at which bonds mature δ plus a further discount to reflect the default survival probability.

4.3 Definition of Equilibrium

We are ready to define an equilibrium:

Definition 2. An equilibrium consists of a compact domain \mathbf{B} and functions of debt, $\{q, V, C\}$, such that: (i) given the government's consumption policy C and strategic default, lenders break even in expectation at prices q ; (ii) given a price schedule q , the government's maximal value conditional on repayment is $V(b)$, which is achieved by consuming $C(b) \in [\underline{C}, \bar{C}]$; and (iii) for $b \in \mathbf{B} = [-\bar{a}, \bar{b}]$, $V(b) \geq \underline{V}$, with $V(\bar{b}) = \underline{V}$.

In the definition of equilibria, we require that $V(\bar{b}) = \underline{V}$. That is, \bar{b} represents the maximal endogenous borrowing limit. We do this to eliminate the possibility of generating equilibria that depend on ad hoc borrowing limits.

¹⁶Condition (v), at \bar{b} , is stronger than necessary, as the key requirement is that $\dot{b} \leq 0$ at the upper bound on debt; however, in the equilibria described below, the stronger condition is always satisfied.

Note that b is the face value of debt, which defines the government's promised payments absent default. The expected present value of payments in equilibrium is the market value of debt: $q(b)b$. This distinction is useful to bear in mind when comparing competitive equilibria to the Pareto problem studied in Section 3.

Mirroring the analysis of efficient allocations, we focus on two types of equilibria. In a *borrowing equilibrium*, the government borrows up to its borrowing limit \bar{b} regardless of initial conditions. In particular, if the government starts in the Safe Zone (or with assets), it borrows into the Crisis Zone and eventually defaults. In a *saving equilibrium*, the Safe Zone is an absorbing state.¹⁷

4.4 The Borrowing Equilibrium

We denote equilibrium objects in the borrowing equilibrium with the subscript B ; that is, C_B, V_B, q_B are the consumption, value, and price functions, respectively. Similarly, let \underline{b}_B denote the threshold between the Safe and Crisis Zones, and \bar{b}_B the endogenous upper bound on debt.

We first state the conjectured borrowing equilibrium and then discuss when it satisfies the equilibrium conditions. In the borrowing equilibrium, we conjecture that the government borrows to its endogenous debt limit. Given the linearity of preferences and weak impatience, a reasonable conjecture is that the government consumes at its upper bound until $b = \bar{b}_B$. At the debt limit, the government pays coupons and rolls over maturing bonds until the first arrival of \bar{V} , at which point it defaults.

Although the borrowing limit is yet to be solved for, we can exploit the analysis of the efficient borrowing allocation to pin down consumption at the limit. Specifically, recall that $C_B^*(\underline{V})$ is the consumption level that delivers value \underline{V} conditional on default when \bar{V} first arrives. Hence, we define

$$C_B(b) \equiv \begin{cases} \bar{C} & \text{if } b \in [-\bar{a}, \bar{b}_B) \\ C_B^*(\underline{V}) & \text{if } b = \bar{b}_B. \end{cases} \quad (23)$$

To complete the definition of C_B , we need to derive \bar{b}_B . We first define

$$\underline{q} \equiv \frac{r + \delta}{r + \delta + \lambda}. \quad (24)$$

This is the break-even price assuming that the government always defaults on the first arrival of

¹⁷In Appendix A, we discuss a third type of Markov equilibrium, which we denote a *hybrid equilibrium* because it combines features of both the saving and borrowing equilibria. Given the multiplicity we discuss below, one could also construct sunspot equilibria.

$$V^D = \bar{V}.$$

At \bar{b}_B , our conjecture is that debt remains constant. From equation (19), the consumption that sets $\dot{b} = 0$ at \bar{b}_B is

$$C_B(\bar{b}_B) = y - (r + \delta)\bar{b}_B + \delta \underline{q} \bar{b}_B. \quad (25)$$

Setting this equal to $C_B^*(\underline{V})$ and rearranging, we have

$$\bar{b}_B \equiv P_B^*(\underline{V})/\underline{q}. \quad (26)$$

The right-hand side is the expected promised payments to the lenders at the efficient borrowing limit divided by the equilibrium price. That is, the market value of debt in the borrowing equilibrium at \bar{b}_B , $\underline{q}\bar{b}_B$, equals the lender's value in the efficient borrowing equilibrium, $P_B^*(\underline{V})$.

The parallelism between the equilibrium and efficient allocation is intuitive in the borrowing scenario. In both allocations, the government consumes the maximal amount until it reaches the indifference point between repaying and defaulting at value \underline{V} .

In particular, given an equilibrium price schedule q_B , we define

$$V_B(b) \equiv V_B^*(q_B(b)b). \quad (27)$$

That is, the government's value at b is the same as in the borrowing planning problem for the associated market value of debt. This equivalence follows from the fact that in both allocations, the government consumes \bar{C} until reaching the limit and then $C_B^*(\underline{V})$ thereafter (until default). In both cases, payments to lenders and default decisions are the same.

This similarity may not be surprising: if the efficient allocation has the government consuming the maximum, this can be implemented in a Markovian equilibrium. However, the close connection between the borrowing equilibrium and the efficient borrowing allocation should not be viewed as suggesting that there is no time consistency problem in the competitive equilibrium. It turns out that there is a major caveat: we show below that the borrowing equilibrium is a competitive equilibrium even in cases when the borrowing allocation is *not* efficient.

Given (23), we can solve for $q_B(b)$ using (22). As the Crisis Zone is an absorbing state in a borrowing equilibrium, we immediately have the equilibrium price

$$q_B(b) \equiv \underline{q} \text{ for } b \in [\underline{b}_B, \bar{b}_B], \quad (28)$$

where we define

$$\underline{b}_B \equiv P_B^*(\bar{V})/\underline{q}. \quad (29)$$

Given the price schedule in the Crisis Zone, we use (22) to extend q_B into the Safe Zone. In particular, setting $\Lambda(b) = 0$ in the safe zone, we differentiate (22) with respect to time to obtain¹⁸

$$\dot{q}_B(b) = q'_B(b)\dot{b} = (r + \delta)q_B(b) - (r + \delta). \quad (30)$$

With the boundary condition $q_B(\underline{b}_B) = \underline{q}$, the solution to this first-order ordinary differential equation (ODE) is defined implicitly by

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}} \right)^{\frac{r}{r+\delta}} = \frac{\bar{C} - y + r q_B(b) b}{\bar{C} - y + r \underline{q} \underline{b}_B}. \quad (31)$$

For each $b \in [0, \underline{b}_B)$, there is a unique solution for $q_B(b) \in [\underline{q}, 1]$. Recall that for $b < 0$, we have $q_B(b) = 1$ regardless of the government's policies.¹⁹

Let $\mathbf{B}_B \equiv [-\bar{a}, \bar{b}_B]$. This completes our conjecture of the borrowing equilibrium, as we have constructed conjectures for the domain, as well as for V_B , C_B , and q_B .

Figure 1 depicts the equilibrium objects for a parameterized borrowing equilibrium. Panel (a) depicts the value function. The dotted horizontal lines represent the two default values, \underline{V} and \bar{V} . The Safe Zone is demarcated by the vertical line at \underline{b}_B . By definition, $V_B(\underline{b}_B) = \bar{V}$ at this point. Similarly, the endogenous upper bound of debt, \bar{b}_B , occurs when $V_B(\bar{b}_B)$ intersects \underline{V} . For reference, the dashed line depicts the value of setting $\dot{b} = 0$, given the equilibrium price schedule and the equilibrium default policy. The stationary value has a discontinuity at \underline{b}_B because defaulting when \bar{V} arrives is strictly better than the stationary value. The stationary value is the same as the equilibrium value at the upper bound \bar{b}_B .

Panel (b) of Figure 1 depicts the price schedule. The price is monotonically decreasing in the Safe Zone and then is flat at \underline{q} for $b \in [\underline{b}_B, \bar{b}_B]$.

The consumption policy function is depicted in Panel (c). For reference, the dashed line depicts the stationary consumption level, given the equilibrium price schedule. Consumption is strictly above the dashed benchmark until $b = \bar{b}_B$, at which point consumption drops to the stationary

¹⁸Note that (22) evaluated at an arbitrary time t is $q(b(t)) = \int_t^\infty e^{-(r+\delta)(s-t)} \int_t^s \Lambda(b(\tau), d\tau [r + \delta] ds$.

¹⁹Note that there may be a discontinuity in q_B at $b = 0$. Recall that at points of discontinuity, we impose that debt buybacks occur at a price of one in the neighborhood around a discontinuity. This restriction eliminates the technical complication of the government attempting to issue debt at one price and near-simultaneously repurchasing at a lower price in an attempt to exploit this discontinuity. The restriction we impose ensures that the choice set is convex despite the discontinuity in price, and hence the government has no motive to "mix" by moving consumption back and forth while keeping debt at the point of discontinuity.

level.

To verify when the conjectured borrowing equilibrium satisfies the equilibrium conditions, we need to check that V_B is a (viscosity) solution of (20). In this case, the important condition is that starting from the Safe Zone, the government prefers to borrow into the Crisis Zone and eventually default rather than remain in the Safe Zone. We have:

Proposition 5. *The conjectured borrowing equilibrium $\{C_B, V_B, q_B, \mathbf{B}_B\}$ is a competitive equilibrium if and only if*

$$V_B(b) \geq \frac{y - [r + \delta(1 - q_B(b))]b}{\rho}, \text{ for all } b \in [0, \underline{b}_B]. \quad (32)$$

The right-hand side of (32) is the value of indefinitely consuming the stationary level of consumption at *equilibrium prices*. Thus, borrowing into the Crisis Zone is an equilibrium outcome if doing so dominates remaining in the Safe Zone.

Crucially, condition (32) is a weaker condition than for borrowing to be efficient, condition (9). As stated in footnote 14, and using that $P = q_B(b)b$, a necessary condition for optimality in the planning problem in the Safe Zone is that

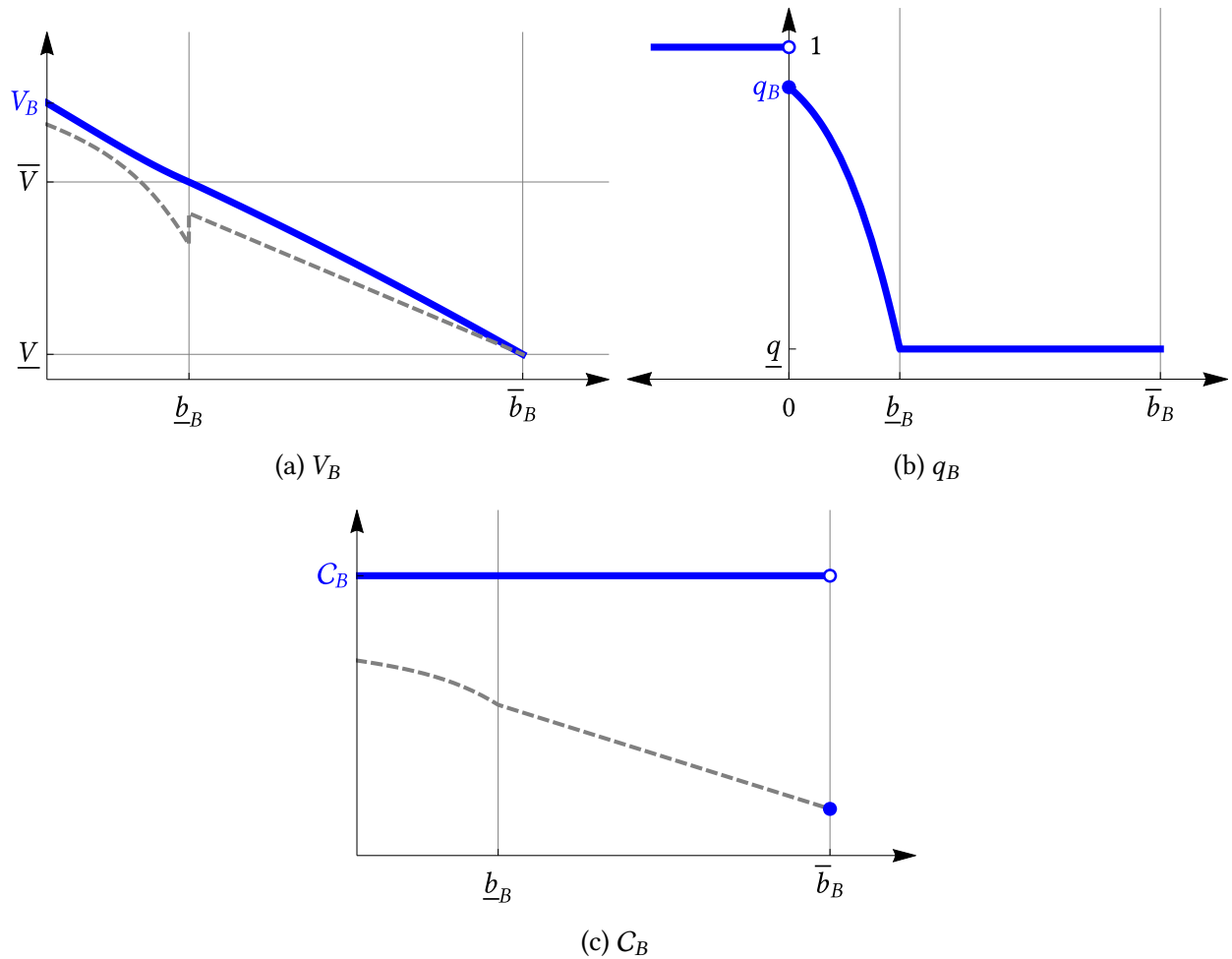
$$V_B(b) = V_B^*(q_B(b)b) \geq \frac{y - rq_B(b)b}{\rho}, \text{ for all } b \in [0, \underline{b}_B]. \quad (33)$$

Both equations (32) and (33) compare the value function to the value that would be generated by keeping the level of debt constant. The difference between equations is the price used to compute this stationary value. In equation (32), the comparison uses the equilibrium prices. In equation (33), the comparison uses the planner's cost of rolling over the lender's value, $q_B(b)b$, at risk-free prices (that is, at an implied interest rate of r). Given that $rq_B(b)b \leq rb \leq [r + \delta(1 - q_B(b))]b$, the competitive equilibrium thus imposes a weaker condition. Hence, *efficiency of borrowing is a sufficient but not necessary condition for the borrowing equilibrium*.

In Figure 2, we plot the market value of debt, $q_B(b)b$, against the corresponding value for the government, $V_B(b)$, using the same parameters as in Figure 1. Specifically, the solid line in the figure depicts the joint surplus between the lenders and the government in a competitive equilibrium. The dashed line is the efficient frontier, which in this parameterization is the efficient saving value, $P_S^*(v)$. The efficient borrowing value, P_B^* , is identical to the equilibrium frontier. The inefficiency of the borrowing equilibrium reflects that the government borrows in the competitive equilibrium, while the planner would like to implement the saving allocation.

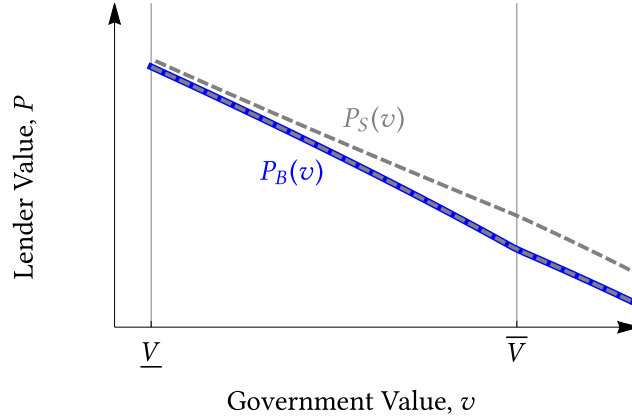
The key difference between the efficiency and equilibrium conditions is that in the latter, the government must pay a default premium reflected in $q_B(b) < 1$. This difference stems from a time consistency problem. In the planning problem, remaining in the Safe Zone is contemplated

Figure 1: Borrowing Equilibrium



The figure depicts the value, price, and consumption functions in a borrowing equilibrium, respectively. The equilibrium functions are represented by the bold solid blue lines. The horizontal lines in the value function plots represent the two default values. The dashed line in the value function plots represents the stationary value function at the corresponding equilibrium prices. The dashed line in the consumption plots represents the level of consumption associated with the stationary value. The equilibrium is constructed with parameters $r = 1$, $\rho = 2$, $y = 1$, $\lambda = 2$, $\delta = 10$, $\bar{C} = 1.2$, $\underline{V} = .8y/\rho$, and $\bar{V} = .95y/\rho$.

Figure 2: Joint Surplus: Borrowing Equilibrium



The figure depicts the joint surplus in the borrowing equilibrium. The solid line is a parametric plot of $(V_B(b), q_B(b)b)$ for $b \in [0, \bar{b}_B]$. The dashed reference line is $P^*(v)$ for $v \in [\underline{V}, V_B(0)]$. The parameters are the same as in Figure 1.

under the assumption of paying the lender a flow r times the value of debt indefinitely, which corresponds to rolling over debt at the risk-free interest rate. As we are in the Safe Zone, this is consistent with the government not defaulting. This may not be a feasible option for the government in a competitive equilibrium. In the borrowing equilibrium, lenders expect that the government in the future will borrow into the Crisis Zone and eventually default. If the government were to remain in the Safe Zone today, *because of these expectations with regard to its future behavior*, the price of the bonds would remain lower than one. Hence, it would nevertheless pay a default premium, rolling debt over at a yield greater than r . Thus, the crucial time consistency problem in the borrowing equilibrium is the inability to credibly commit not to exit the Safe Zone at some point in the future. The link between creditor beliefs about future fiscal policy and the government's best response to the resulting equilibrium price schedule will provide the source of multiplicity discussed in the next section.

Maturity is at the heart of this time consistency problem. To see this, let us consider what happens when $\delta \rightarrow \infty$, that is, as the bonds mature instantaneously (the appropriate continuous time analog of one-period debt). In the proof of the next proposition, we show that $q_B(b) \rightarrow 1$ and $\delta(1 - q_B(b)) \rightarrow 0$ for $b \in [0, \underline{b}_B)$, as $\delta \rightarrow \infty$. Hence, the equilibrium condition (32) and the efficiency condition (33) become identical.²⁰ More generally, the proof of the following proposition establishes that condition (32) becomes stronger as δ increases. Summarizing the above,

Proposition 6. *The following holds:*

- (i) *If the borrowing allocation is efficient, then the conjectured borrowing equilibrium is a compet-*

²⁰Note that δ has no effect on V_B^* because the planning problem is independent of maturity. Even though \underline{b}_B is affected by changes in δ , $q_B(\underline{b}_B)\underline{b}_B$ remains constant.

itive equilibrium for any δ ;

(ii) If the borrowing equilibrium exists for δ_0 , then it exists for any $\delta \in [0, \delta_0]$; and

(iii) If the borrowing allocation is not efficient, then there exists a $\delta_1 < \infty$ such that the conjectured borrowing equilibrium is not a competitive equilibrium for $\delta > \delta_1$.

4.5 The Saving Equilibrium

We now consider an alternative equilibrium that features saving out of the Crisis Zone. As in the efficient saving allocation, we conjecture that the Safe Zone is an absorbing state and the Crisis Zone can potentially be divided into a saving region and a borrowing region. Let V_S^* denote the inverse of the efficient saving solution, P_S^* . Let $(-\bar{a}, \underline{b}_S]$ denote the Safe Zone, $(\underline{b}_S, b^I]$ the saving region in the Crisis Zone, and $(b^I, \bar{b}_S]$ the borrowing region in the Crisis Zone. Let $\{\hat{V}, \hat{C}, \hat{q}\}$ denote the conjectured equilibrium objects in the saving region. We define these objects below. The functions $\{V_B, q_B, C_B\}$ correspond to the borrowing equilibrium defined in the previous subsection.

The conjecture for the saving equilibrium is

$$V_S(b) \equiv \begin{cases} V_S^*(b) & \text{for } b \in [-\bar{a}, \underline{b}_S] \\ \hat{V}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ V_B(b) & \text{for } b \in (b^I, \bar{b}_S]; \end{cases} \quad (34)$$

and

$$q_S(b) \equiv \begin{cases} 1 & \text{for } b \in [-\bar{a}, \underline{b}_S] \\ \hat{q}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ \underline{q} & \text{for } b \in (b^I, \bar{b}_S]; \end{cases} \quad (35)$$

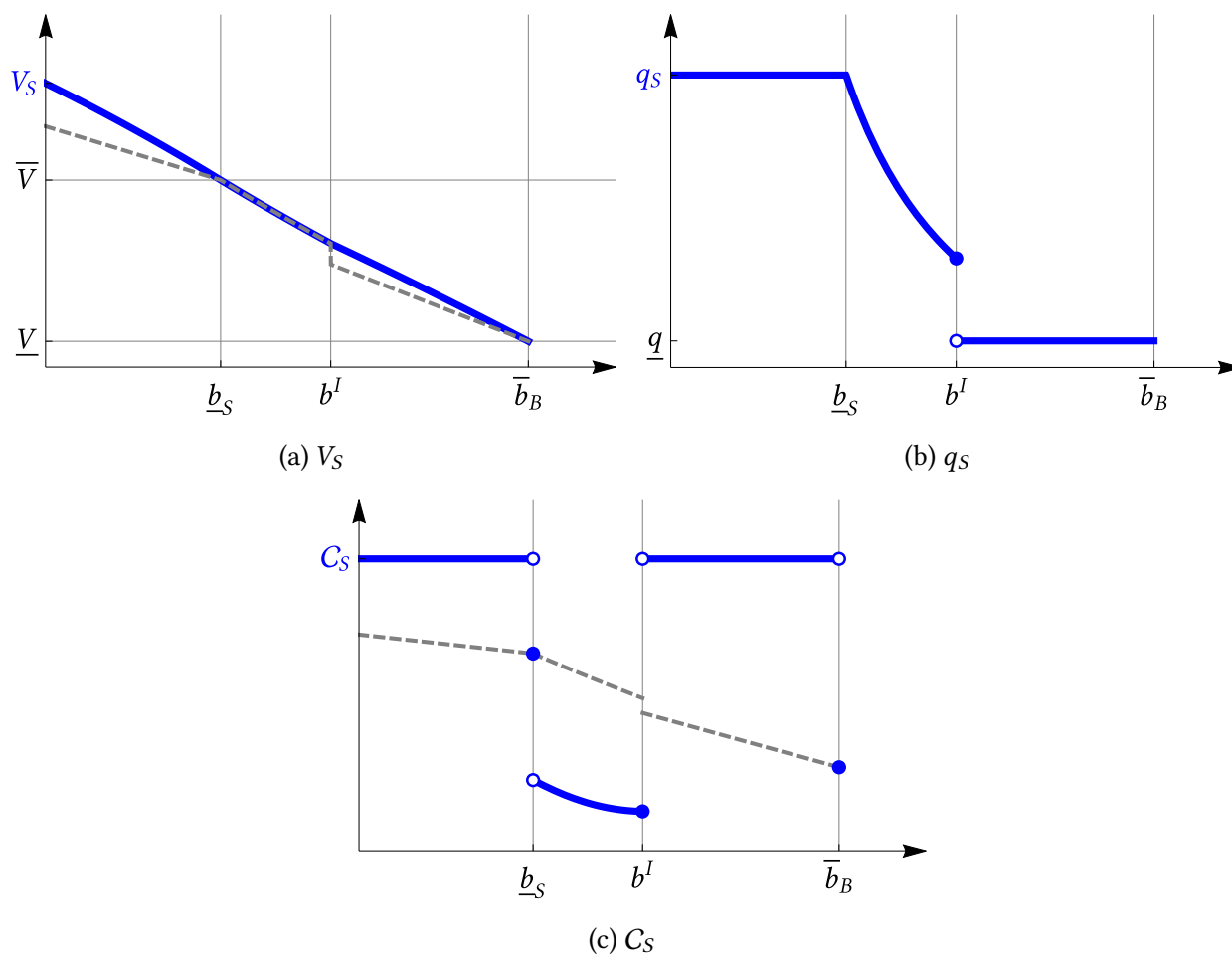
and

$$C_S(b) \equiv \begin{cases} C_S^*(V_S^*(b)) & \text{for } b \in [-\bar{a}, \underline{b}_S] \\ \hat{C}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ C_B(b) & \text{for } b \in (b^I, \bar{b}_S]. \end{cases} \quad (36)$$

The equilibrium objects are depicted in Figure 3, which follows the layout of Figure 1.

The Safe Zone is straightforward: because it is an absorbing region, there is no risk of default starting from $b \leq \underline{b}_S$. Hence, the price is one and the values and consumption are equivalent to

Figure 3: Saving Equilibrium



The figure depicts the value, price and consumption functions in a saving equilibrium, respectively. The equilibrium functions are represented by the bold solid blue lines. The horizontal lines in Panel (a) represent the two default values. The dashed line in Panel (a) represents the stationary value function at the corresponding equilibrium prices. The dashed line in Panel (b) represents the level of consumption associated with the stationary value. The equilibrium is constructed with the same parameters as Figure 1: $r = 1$, $\rho = 2$, $y = 1$, $\lambda = 2$, $\delta = 10$, $\bar{C} = 1.2$, $\underline{V} = .8y/\rho$, and $\bar{V} = .95y/\rho$. The value of \underline{C} is set low enough so that it never binds in equilibrium.

their efficient counterparts. Similarly, the boundary of the Safe Zone parallels that of the planning problem; that is, \underline{b}_S is the level of debt that delivers value \bar{V} when debt is constant and the price is one:

$$\underline{b}_S \equiv P_S^*(\bar{V}) = \frac{y - \rho\bar{V}}{r}. \quad (37)$$

At the boundary of the Safe Zone, V_S and C_S equal the stationary values and consumption, respectively.

The borrowing region of the Crisis Zone is also an absorbing state and corresponds to the equilibrium discussed in the previous subsection. Note that in this region, the price is \underline{q} .

The final step is to characterize the saving region of the Crisis Zone as well as the boundaries $\{b^I, \bar{b}_S\}$. In the saving region, we have to deviate from the prescription of the efficient allocation. The reason is that the efficient savings policy, which sets consumption at its lower bound \underline{C} , cannot be sustained in a competitive equilibrium. That is, the efficient savings rate is not privately optimal in an equilibrium with long-term bonds. We provide intuition for this result below.

We conjecture instead that the government saves by consuming at an interior optimum.²¹ When consumption is interior, the linearity of the government's objective function in (20) implies that it is indifferent across alternative consumption choices, including the consumption level that sets $\dot{b} = 0$. Hence, the government must be indifferent between the equilibrium consumption strategy and its associated stationary value.²²

$$\hat{V}(b) \equiv \frac{y - [r + \delta(1 - \hat{q}(b))]b + \lambda\bar{V}}{\rho + \lambda}. \quad (38)$$

Interior consumption requires $\hat{V}'(b) = -\hat{q}(b)$. Using this, differentiating (38), and solving the resulting ODE with $\hat{q}(\underline{b}_S) = 1$ as a boundary condition yields

$$\hat{q}(b) \equiv \frac{r + \delta + \left(\frac{b}{\underline{b}_S}\right)^{-\frac{\rho+\lambda+\delta}{\delta}} (\lambda + \rho - r)}{\rho + \lambda + \delta}. \quad (39)$$

The lenders' break-even condition (22) requires $\hat{q}'(b)\dot{b} = (r + \delta + \lambda)\hat{q}(b) - (r + \delta)$. Hence, we can

²¹Throughout the following analysis, we assume \underline{C} is sufficiently low that an interior consumption choice is feasible.

²²The fact that the government's value is equal to the stationary value while consumption is interior is discussed in Tourre (2017) and DeMarzo et al. (2018). The authors give an interpretation of a durable monopolist in the spirit of the Coase conjecture.

solve for the conjectured debt dynamics:

$$\dot{b} = -\delta b \left(\frac{\hat{q}(b) - \underline{q}}{\hat{q}(b) - \underline{q} + \frac{(\rho-r)\hat{q}(b)}{r+\delta+\lambda}} \right) \equiv f(b). \quad (40)$$

Using (19), we obtain

$$\hat{C}(b) \equiv y - [r + \delta(1 - \hat{q}(b))]b + \hat{q}(b)f(b). \quad (41)$$

In the Crisis Zone, $V_S(b) = \max\langle \hat{V}(b), V_B(b) \rangle$. As before, b^I is the intersection point of these two alternatives. If no such $b^I \in [\underline{b}_S, \bar{b}_B]$ exists, we set it to \bar{b}_S . The value of \bar{b}_S is such that $V_S(\bar{b}_S) = \underline{V}$, and we define $\mathbf{B}_S \equiv (-\bar{a}, \bar{b}_S]$.²³

As in our discussion of efficient allocations, the question is whether it is optimal to remain in the Safe Zone or borrow to the upper bound. Crucially, for the equilibrium, the question is now whether the government finds it *privately* optimal. The condition for saving to be a valid equilibrium outcome is stated in the following proposition:

Proposition 7. *The conjectured saving equilibrium $\{C_S, V_S, q_S, \mathbf{B}_S\}$ is a competitive equilibrium if and only if*

$$V_S(\underline{b}_S) \geq V_B(\underline{b}_S); \quad (42)$$

or, equivalently,

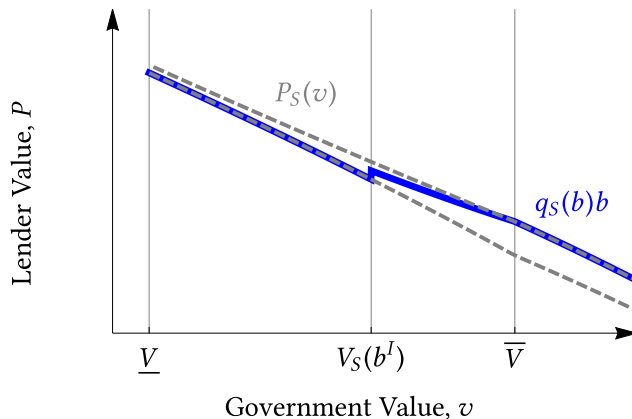
$$\underline{b}_S \equiv \frac{y - \rho \bar{V}}{r} \geq \underline{b}_B. \quad (43)$$

To see why saving can be an equilibrium outcome, first note that the government always has the option to remain in the Crisis Zone and wait for the high default option. As $\bar{V} > V_S(b)$ in the Crisis Zone, this is a plausible alternative. The cost of this strategy is that the government must roll over its debt at a discounted price while waiting for \bar{V} . If instead the government saves to the Safe Zone, it can roll over its debt at the risk-free price. This increase in price ensures that the government internalizes the gain from reducing the probability of default and provides the government with the incentive to save.

However, the government's private incentive to save in equilibrium is weaker than that of

²³ If $b^I < \bar{b}_B$, then q_S is discontinuous at b^I , which is the case depicted in Figure 3. As previously discussed when stating the government's problem, and echoed in footnote 19, we rule out the government issuing at $q_S(b^I)$ and then immediately repurchasing at $\lim_{b' \downarrow b^I} q_S(b') < q_S(b^I)$ in an attempt to set $\dot{b} = 0$ by alternating between issuing and repurchasing. Let us also note that the multiplicity result we obtain later on does not hinge on this particular issue: it is possible to obtain parameter values such that $b^I = \bar{b}_S$ and for which multiple equilibria coexist.

Figure 4: Joint Surplus: Saving Equilibrium



The figure depicts the joint surplus in the saving equilibrium. The solid line is a parametric plot of $(V_S(b), q_S(b)b)$ for $b \in [0, \bar{b}_B]$. The upper and lower dashed reference lines are $P_S^*(v)$ and $P_B^*(v)$, respectively, for $v \in [\underline{V}, V_B(0)]$. The parameters are the same as in Figure 1.

the planner. We see this in two ways. First, recall from Proposition 3 that saving is a solution to the planning problem if $(y - \rho\bar{V})/r \geq P_B^*(\bar{V}) = \underline{q}\underline{b}_B$.²⁴ As $\underline{q} < 1$, condition (43) is *stronger* than the efficiency condition. Thus, *efficiency of saving does not imply that it can be sustained in equilibrium*. That is, a necessary but not sufficient condition for a saving equilibrium to exist is that the saving allocation is efficient.

Second, even when the government saves, it does not do so at the same rate as the planner. In the debt dynamics equation (40), the term in parentheses is less than one in magnitude, and thus $\dot{b} \geq -\delta b$. That is, while saving, the government never repurchases non-matured bonds; it deleverages by letting bonds mature and not fully replacing them with new bonds. This reflects the inefficiency of long-term debt discussed by Aguiar et al. (2018). The government does not capture the full return to eliminating the probability of default and thus does not have an incentive to save as quickly as possible. This leads to a divergence between the saving equilibrium allocation and the efficient saving allocation.

Figure 4's solid line plots the market value of debt, $q_S(b)b$, against the government's value, $V_S(b)$. The upper and lower dashed lines are the efficient frontier for the saving and borrowing allocation, respectively. The saving allocation dominates the borrowing allocation and hence represents the Pareto frontier. For $v \in [\underline{V}, V_S(b^I)]$, that is, for $b \in (b^I, \bar{b}_B]$, the government borrows when it is efficient to save. For $v \in [V_S(b^I), \bar{V}]$, or $b \in [\underline{b}_S, b^I]$, the government saves, but at a rate that is inefficiently slow. Hence the equilibrium surplus remains within the Pareto frontier. Note that the discontinuity in the equilibrium price schedule at b^I is reflected in the sharp change in the lender's value around this threshold. For $v \geq \bar{V}$, or $b \leq \underline{b}_S$, the government is in the Safe

²⁴The latter equality uses the fact that $V_B(\underline{b}_B) = V_B^*(\underline{q}\underline{b}_B) = \bar{V}$ and V^* is the inverse of P^* .

Zone, and the efficient and equilibrium allocations coincide.

The gap between the efficient allocation and the equilibrium outcome highlights the weak incentives provided by prices when bonds have longer maturity. Recall that in the saving region, $b \in (\underline{b}_S, b^I)$, the government is indifferent across consumption choices given equilibrium prices. The purchasers of new bonds break even and are also indifferent. However, the legacy bondholders strictly prefer $c = \underline{C}$, as this minimizes the risk of default going forward (and maximizes the secondary market value). However, there is no market mechanism that ensures the government internalizes the interests of legacy bondholders, leading to the inefficiency depicted in Figure 4 on the domain $v \in [V_S(b^I), \bar{V}]$.²⁵

Relatedly, the difference between efficiency and equilibrium, and the private incentives to save, depends on maturity. In particular, the greater the fraction of debt rolled over each period, the stronger the government's private incentive to save, while maturity is irrelevant for the efficient allocation. At one extreme, if $\delta = 0$ and bonds are perpetuities, the government never saves in equilibrium regardless of efficiency; at the other extreme, as $\delta \rightarrow \infty$, the conditions for saving to be efficient and to be an equilibrium outcome converge. Collecting results:

Proposition 8. *A necessary condition for $\{C_S, V_S, q_S, \mathbf{B}_S\}$ to be a competitive equilibrium is that saving is efficient. If saving is strictly efficient, that is, $P_S^*(\bar{V}) > P_B^*(\bar{V})$, there exists a $\delta_S \in [0, \infty)$, defined by*

$$\delta_S \equiv \frac{\lambda P_B^*(\bar{V})}{P_S^*(\bar{V}) - P_B^*(\bar{V})} - r, \quad (44)$$

such that $\{C_S, V_S, q_S, \mathbf{B}_S\}$ is a competitive equilibrium if $\delta \geq \delta_S$, and is not an equilibrium otherwise. If $\rho > r$, then $\delta_S > 0$.

The fact that maturity drives a wedge between efficiency and equilibria anticipates the next section. Even when saving is efficient and can be supported as an equilibrium, it is still possible that the borrowing allocation remains a valid competitive equilibrium. In the next section, we discuss the role of maturity in this multiplicity.

5 Maturity and Multiplicity

The preceding section provided necessary and sufficient conditions for both the borrowing and saving equilibria. This allows us to explore under what parameterizations the model has multiplicity as well as the economics behind the multiplicity.

²⁵As shown in Aguiar et al. (2018), the inefficiency of the competitive equilibrium with longer-term debt survives when the government has the ability to replace long-term bonds with short-term debt through competitive market transactions.

The key condition to sustain either equilibrium is whether the government prefers to remain in the Safe Zone or borrow into the Crisis Zone. Importantly, the government makes this decision taking the equilibrium price schedule as given. This is the crucial distinction between the equilibrium problem and the planning problem and is at the heart of the potential multiplicity.

First, consider the borrowing equilibrium depicted in Figure 1. While in the Safe Zone ($b < \underline{b}_B$), there is no threat of immediate default as $V_B(b) \geq \bar{V}$. Nevertheless, the bond price lies strictly below one. The creditors require a default premium because they anticipate that the government will borrow into the Crisis Zone ($b > \underline{b}_B$), and then potentially default, before the debt matures. Hence, the government does not have the option to remain in the Safe Zone at risk-free prices. Rather, the question is whether to maintain its debt position in the Safe Zone at a price below one, or borrow into the Crisis Zone. As can be seen, the stationary value in the Safe Zone lies strictly below the equilibrium value function. Given that the price schedule offers no reward for remaining in the Safe Zone, the creditors' pessimistic expectations become self-fulfilling.

Now consider the saving equilibrium depicted in Figure 3, constructed with the same parameter values. Note that the equilibrium price is one throughout the Safe Zone and then declines in the Crisis Zone. This nonlinearity in the price schedule is reflected in the government's value function. The payoff to saving out of the Crisis Zone is the high price at the boundary of the Safe Zone.

Interestingly, across the two equilibria, the government borrows when prices are low (spreads are high), while it saves when prices are high (spreads are low). The important element of the price schedule is not the level, but the incentives or disincentives to borrow. In the saving equilibrium, the price schedule declines steeply once the government enters the Crisis Zone. In the borrowing equilibrium, the price schedule is flat at the boundary of the Safe Zone. In this way, the self-fulfilling dynamics we uncover in this paper provide an alternative view of the "gambling for redemption" hypothesis that explains the debt accumulation of debt-distressed European countries during the debt crises (see Conesa and Kehoe, 2017). In our model, low debt prices and debt accumulation both arise endogenously.

Note that the multiplicity in the model is *dynamic* in that it depends on expectations of future equilibrium behavior. In particular, the equilibria are supported by different expectations about whether the government will borrow or save, and whether bond prices will be the risk-free price or something lower. The underlying tension is between the incentive to dilute long-term bondholders versus the incentive to economize on rollover costs. Which effect dominates in equilibrium depends on beliefs in a non-trivial part of the parameter space. Moreover, these competing forces highlight why maturity plays a central role in the existence of multiple equilibria.

For the limiting case of arbitrarily large \bar{C} , we can state a simple condition that determines when it is possible for both equilibria to be supported:

Proposition 9. *If the parameters satisfy the following condition:*

$$1 + \rho \left(\frac{\bar{V} - V}{y - \rho \bar{V}} \right) > \frac{\lambda}{\rho - r} > r \left(\frac{\bar{V} - V}{y - \rho \bar{V}} \right), \quad (45)$$

there exists an M and a non-empty interval $\Delta \subset [0, \infty)$, such that for all $\bar{C} > M$ and all $\delta \in \Delta$, both the borrowing and saving equilibria exist.

The second inequality in (45) guarantees that the saving allocation is efficient for arbitrarily large \bar{C} . We know from Proposition 8 that this is a necessary condition and sufficient for high enough δ for the saving equilibrium to exist.

The first inequality in (45) guarantees the existence of the borrowing equilibrium, for any finite δ , when \bar{C} becomes arbitrarily large. When \bar{C} becomes arbitrarily large, the price of the bond converges to \underline{q} throughout the Safe Zone, as the rate at which the government exits the Safe Zone becomes arbitrarily fast. The first inequality verifies that the government prefers to borrow into the Crisis Zone when facing a price close to \underline{q} for all debt levels in the Safe Zone.

This proposition shows that multiplicity is an endemic feature of this model when the government is impatient and there are deadweight losses from default. That is,

Corollary 3. *If $\rho > r$ and $y > \rho \bar{V}$, there always exists a triplet $\{\delta, \lambda, \bar{C}\}$ such that both savings and borrowing equilibria exist.*

6 Relationship to other sources of multiplicity

The core driving force behind the multiplicity is the feedback between the price schedule and the government's debt-issuance policy function. In particular, the incentives embedded in equilibrium prices to alter the outstanding stock of debt before legacy bonds mature. It is useful to contrast this mechanism with other environments that feature multiplicity.

The source of multiplicity studied above is distinct from the canonical Calvo (1988) multiplicity. A useful way to view the Calvo multiplicity is through the feasibility of debt trajectories. That is, the mechanical link via the budget set between low prices of bond issuances today and high debt burdens tomorrow. In the two-period model of Calvo (1988), today's bond prices (or the implied interest rate) determine the debt burden tomorrow, given the requirement to raise a certain amount of revenue in the initial period. Lorenzoni and Werning (2013) (henceforth, LW) explore how to extend this mechanism to a fully dynamic model.

LW emphasize that, in practice, governments have limited flexibility to alter fiscal policy at high frequencies. Most of their analysis therefore assumes the government follows a fiscal rule. However, they show that there is a natural counterpart to a fiscal rule in an environment with

a fully optimizing government. In particular, they consider a government making optimal debt decisions subject to a minimum threshold for spending (which in their case arises naturally from the non-negativity of spending).

LW multiplicity turns on whether the government is *able* to reduce its debt when facing low bond prices. In particular, they consider a situation where the bond price may be low enough that it is not feasible for the government to reduce debt. As a result, the government optimally chooses to accumulate debt instead, justifying the low bond prices. Importantly, they show how such a feasibility constraint on expenditure can generate multiplicity even though it does not bind on the equilibrium path – once the governments decides to borrow, its expenditures are not constrained by the lower bound. But, as noted by LW, the potentially binding constraint is necessary to generate the multiplicity they study.

It is useful to highlight how the LW multiplicity differs from the one in our analysis. The equivalent of a minimum spending threshold in our environment is the consumption lower bound \underline{C} . In our equilibria, we assume that this lower bound does not bind.²⁶ Moreover, even when facing the “borrowing” equilibrium prices, we implicitly assume that it is always *feasible* but not optimal for the government to save towards the Safe Zone.

This begs the question of whether the LW multiplicity arises if the consumption lower bound \underline{C} is high enough. To answer this, let us now attempt to construct the LW multiplicity in our environment. First, consider parameters, including \underline{C} , such that the saving equilibrium exists. To construct an alternative LW “borrowing” equilibrium, suppose that there exists a $b_0 \in (\underline{b}_S, \bar{b}_B)$ such that reducing the debt is *not feasible at the borrowing equilibrium prices, \underline{q}* .²⁷ That is,

$$\underline{C} > y - (r + \delta(1 - \underline{q}))b_0. \quad (46)$$

This condition implies that, at a price of \underline{q} and $b \geq b_0$, even if the government were to set its consumption to its lower bound, debt still strictly increases. Such a restriction is used by LW to sustain an alternative equilibrium for high enough debt levels: given that the government cannot reduce debt, it is now willing to follow the borrowing equilibrium prescription and accumulate debt until \bar{b}_B (justifying the price of \underline{q}).

However, in our environment, such an equilibrium cannot exist. To see this, if equation (46) holds, Assumption 1 (iv) implies that $b_0 > \bar{b}_B$.²⁸ The reason is that if $\dot{b} \leq 0$ is not feasible at \underline{q}

²⁶We introduced a lower bound on consumption only to have a well defined policy for the efficient allocation.

²⁷Note that, the savings equilibrium is not necessarily affected by the lower bound. That is, it could be the case that reducing the debt remains feasible at high prices. A sufficient condition in our case is that $\hat{C}(b) = y - (r + \delta(1 - \hat{q}(b))b + \hat{q}(b)f(b) > \underline{C}$ for $b \in (\underline{b}_S, b_0)$.

²⁸A remaining issue is what would happen if we were to drop Assumption 1 (iv) from our requirements. In that case, in the borrowing allocation, the maximal amount of debt, \bar{b}_B that could be feasibly sustained in a borrowing equilibrium is such that $\underline{C} = y - (r + \delta(1 - \underline{q}))\bar{b}_B$. Inequality (46) still implies $b_0 > \bar{b}_B$.

and $b = b_0$, then $\dot{b} > 0$ for all $b > b_0$, including the upper-bound \bar{b}_B , violating the equilibrium conditions.²⁹

Interestingly, while imposing a tight lower bound on consumption in our environment does not generate the equilibrium studied by LW, it can produce a roll-over crisis a la Cole and Kehoe (2000) (henceforth CK). Consider again a situation in which savings is an equilibrium. We ask the question of whether we can also sustain an equilibrium where the bond price switches from the savings equilibrium price to a price of zero within a subset of the domain $(\underline{b}_S, \bar{b}_S)$.

We conjecture the following “failed-auction” equilibrium: there exists a $b_0 \in (\underline{b}_S, \bar{b}_S)$ and an equilibrium price q such that $q(b) = 0$ for $b > b_0$, and $q(b) = q_S(b)$ otherwise, where q_S is the saving equilibrium price schedule. Note that this implies $V_S(b_0) > \underline{V}$.

First, suppose $\underline{C} < y - (r + \delta)b$ for $b \in (\underline{b}_S, \bar{b}_S)$. This implies that it is feasible for the government to reduce debt by paying coupons and maturing bonds. Now consider a b in the neighborhood above b_0 . The government has the option to reduce debt to b_0 by paying off maturing bonds. The value of this strategy is arbitrarily close to $V_S(b_0)$ as $b \downarrow b_0$. As $V_S(b_0) > \underline{V}$, the government will not find it optimal to default in this neighborhood, invalidating zero as an equilibrium price. With no lower bound on consumption, the government can always act as its own lender of last resort in case of a rollover crisis, eliminating the CK multiplicity.

Alternatively, continuing the premise that the saving equilibrium exists, suppose \underline{C} satisfies the constraint (46) for a $b_0 \in (\underline{b}_S, \bar{b}_S)$.³⁰ For $b > b_0$, there is no feasible option for the government other than default. This follows from the fact that (46) implies the government cannot pay off its coupon and maturing principal payments at $b = b_0$ when the price is $\underline{q} > 0$; hence, it cannot do so at $b > b_0$ when facing a zero price. Thus, the government must default, validating the zero price in equilibrium.

In summary, the difference between the equilibria we study and those of Calvo, CK, and LW reflect different views with regards to fiscal policy. This paper emphasizes the *lack* of constraints on fiscal policy, bringing limited commitment to future fiscal paths to the fore of the analysis. Calvo, CK, and LW emphasize the *limitations* of fiscal policy when responding to low bond prices. Both views are complementary, and highlight the potential fragility of sovereign debt markets to changes in lender expectations.

²⁹Differently from us, LW assume that there is a positive recovery rate upon default and impose an upper-bound on debt, at which point, a renegotiation between creditors and the government is automatically triggered. This upper-bound guarantees that debt does not grow without bounds in the “bad” equilibrium, even if it cannot be kept stationary absent renegotiation.

³⁰Note that this implies from Assumption 1 (iv) that $b_0 > \bar{b}_B$.

7 Third-Party Policies

The existence of multiple equilibria raises the question of whether a deep-pocketed third party, such as the IMF or ECB, could induce market participants to play the preferred equilibrium. In the rollover crisis model of Cole and Kehoe (2000), a price floor would eliminate the crisis equilibrium. Similarly, in a Calvo-style crisis, a price floor (or a cap on spreads) would also eliminate the bad equilibrium. More importantly, such a policy would require no resources along the equilibrium path, as long as they were credible off equilibrium.

A natural policy question in our framework is how to prevent coordination on the borrowing equilibrium when saving is efficient. Debt forgiveness does not select a particular equilibrium because both equilibria co-exist at low debt levels. Hence, in the borrowing equilibrium, debt forgiveness provides only a temporary reduction in debt levels, as in the debt-overhang model of Aguiar and Amador (2011). Similarly, a price floor does not eliminate the inefficient equilibrium. In particular, with a lower bound on prices greater than \underline{q} , the borrowing equilibrium remains an equilibrium and the government would borrow up to its borrowing limit at the better price. The policy not only would fail, but also would cost resources along the equilibrium path.

More formally, consider a parameterization such that both saving and borrowing equilibria exist, with subscripts B and S denoting the respective equilibrium objects, as before. This parameterization is the natural launching point for policy intervention.

The intervention we study involves a third party that is willing to purchase government bonds at a price q^* as long as $b \leq b^*$. This combines a price floor with a quantity restriction. To highlight the role of the price floor versus the quantity restriction, we consider two polar cases. In our first scenario, let $b^* = \bar{b}_B$. That is, the quantity restriction is not tighter than the endogenous borrowing limit in the borrowing equilibrium. The second scenario sets $b^* = \underline{b}_S$. This is a tight quantity restriction, designed such that interventions potentially involve only risk-free debt.

Let the superscript P indicate equilibrium objects in the presence of the third-party policy. The break-even condition for foreigners is

$$q^P(b) = \sup_{T \geq 0} \left\{ \int_0^T e^{-(r+\delta)t - \int_0^t \Lambda^P(b^P(s)) ds} (r + \delta) dt + e^{-(r+\delta)T} \mathbb{1}_{[b^P(T) \leq b^*]} q^* \right\}, \quad (47)$$

where $b^P(s)$ denotes the equilibrium evolution of bonds, starting from b , under the third-party policy. The equation captures that an investor considers the best among all possible hold-and-sell strategies: after purchasing the bonds, the investor can hold them up to any time T , at which point, if the total debt remains below b^* , the investor has the option to sell them to the third party for a price of q^* . Note that the assumption that all the investors are identical means we do not need to consider the strategies where one investor sells to another.

Given the price schedule, the problem of the government continues to be characterized by the HJB (20). As a result, in any equilibrium, there will be a Safe Zone and a Crisis Zone, demarcated by $\{\underline{b}^P, \bar{b}^P\}$, with $V^P(\underline{b}^P) = \bar{V}$ and $V^P(\bar{b}^P) = \underline{V}$.

As in the analysis without the third party, we will consider two equilibrium conjectures: a borrowing one and a saving one. Similarly to our benchmark analysis, in a conjectured borrowing equilibrium, starting from a debt level in the Safe Zone, the debt eventually reaches the Crisis Zone. In a conjectured saving equilibrium, the Safe Zone is an absorbing state.

Consider first the case where $b^* = \bar{b}_B$. In this case, the policy does not eliminate the borrowing equilibrium. But if it is generous enough (that is, if q^* is high enough), then it eliminates the saving equilibrium:

Proposition 10 (Loose quantity restriction). *Assume the inequalities in Proposition 9 are satisfied and $\bar{b}_B > \underline{b}_S$. Suppose $q^* \in (\underline{q}, 1]$ and $b^* = \bar{b}_B$, and let \bar{C} be sufficiently large. Then,*

(i) *There always exists a borrowing equilibrium. That is, there is an equilibrium where $C^P(b) = \bar{C}$ for all $b < b^*$. In this equilibrium, the third party incurs losses.*

(ii) *There is a $\tilde{q} < 1$ such that for all $q^* > \tilde{q}$, the saving equilibrium does not exist.*

A better policy is to impose a tighter quantity restriction, that is, $b^* = \underline{b}_S$. In this case, the policy selects the saving equilibrium for high enough q^* :

Proposition 11 (Tight quantity restriction). *Assume the inequalities in Proposition 9 are satisfied. Suppose $q^* \in [\underline{q}, 1]$ and $b^* = \underline{b}_S$. Then,*

(i) *The saving equilibrium is always an equilibrium. The third party incurs zero losses.*

(ii) *There is a $\hat{q} < 1$ such that for all $q^* > \hat{q}$, the borrowing equilibrium does not exist.*

The propositions above show that a price floor policy has very different implications, depending on the quantity restriction that accompanies it. If the quantity restriction is loose, a generous price floor ends up incentivizing borrowing and generates losses for the third party. However, if the quantity restriction is tight enough, a generous price floor eliminates the sub-optimal borrowing equilibria, and no resources are lost by the third party on equilibrium. In fact, in the latter case, the third party never needs to purchase debt in equilibrium.

Recall that the multiplicity reflects the trade-off between saving for a better price versus the desire to borrow due to impatience. With a price floor absent a tight quantity restriction, the third party reduces the incentive to save. The saving equilibrium is supported by the gap between prices in the Safe Zone and prices in the Crisis Zone as well as the need to roll over bonds. A generous price floor in the Crisis Zone eliminates the price differential that incentivizes saving in equilibrium.

Rewarding the government for saving, or punishing them for borrowing, is a policy that can induce the saving equilibrium. A borrowing limit at the boundary of the Safe Zone, which is tighter than the endogenous limit, would be effective. However, such a policy raises the question of how to enforce the limit if the initial debt is beyond it. Third-party purchases conditional on fiscal austerity are reminiscent of policies pursued in the European debt crisis as well as many IMF programs. However, the events in Europe and elsewhere reflect the difficulties of enforcing explicit debt limits. Unfortunately, in the Eaton-Gersovitz framework studied in this paper, there is no effective policy that does not involve a similar type of off-equilibrium commitment to punish overborrowing.

Finally, note that a tight quantity restriction policy may not be effective if delayed too long. In particular, once $b > b^l$, the saving equilibrium is no longer distinguishable from the borrowing equilibrium, and thus policy interventions will fail to be effective once debt has reached sufficiently high levels. This highlights that interventions during debt crises may need to be quick to be successful, and policies that “kick the can down the road” may eventually fail. This same point about delay, although in a different environment, was emphasized by Lorenzoni and Werning (2013).

8 Conclusion

This paper shows that debt dilution generates multiplicity in a standard sovereign debt framework. In particular, the extent of dilution in equilibrium depends on self-fulfilling expectations of future prices and future fiscal policy. A relatively impatient government, an intermediate debt maturity, and deadweight losses from default provide the conditions for multiplicity of equilibria. Importantly, these are common features of observed debt markets as well as the recent quantitative models proposed in the literature.

The framework presented above is designed for analytical clarity and thus involves some special assumptions. However, the mechanism at work is robust to including endowment fluctuations and risk aversion, which, while bringing the model closer to empirical debt markets, does not eliminate the self-fulfilling debt dilution identified in the tractable model. One can easily construct simple numerical examples of multiplicity with these elements. The quantitative model analyzed by Stangebye (2015) also appears to be driven by a mechanism similar to that studied in this paper. Indeed, the fact that the Eaton-Gersovitz model is vulnerable to dilution is at the heart of the recent quantitative literature that attempts to match empirical sovereign debt crises. We show that the same force leads to indeterminacy. The fact that multiplicity stems from the incentives to dilute places novel restrictions on effective third-party interventions.

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Appendix A: The Hybrid Equilibrium

In this appendix, we present a third type of competitive equilibrium, which we label the “hybrid” equilibrium because it combines features of both borrowing and saving equilibria. In particular, the government never saves, as in the borrowing equilibrium, but part of the Safe Zone is absorbing, as in the saving equilibrium. The main purpose of introducing the hybrid equilibrium is to show existence of a competitive equilibrium; in particular, we prove that if neither the borrowing nor the saving equilibrium exists, then a hybrid equilibrium exists. The equilibrium objects are depicted in Figure A.1 using the same parameters as in Figures 1 and 3.

More formally, given V_B in (27), define the threshold

$$V_B(b_H) = \frac{y - rb_H}{\rho}, \quad (48)$$

if such a threshold exists on the domain $[0, \underline{b}_B] \cap [0, \underline{b}_S]$. The equilibrium conjecture is that for $b \leq b_H$, the government borrows up to b_H and then remains there indefinitely. This behavior is similar to the Safe Zone policy in the saving equilibrium, but the threshold b_H may be strictly below \underline{b}_S . At b_H , given that $V_B(\underline{b}_H) = (y - rb_H)/\rho$, the government is indifferent to remaining at b_H at risk-free prices versus borrowing to the debt limit at the borrowing equilibrium price schedule. The conjecture is that for $b > b_H$, the government borrows. In a hybrid equilibrium, therefore, b_H is a stationary point that is stable from the left but not the right.

For $b < b_H$, we solve the government’s HJB assuming $c = \bar{C}$ to obtain a candidate V_H on this domain, using the boundary condition $\rho V_H(b_H)y - rb_H$. For $b > b_H$, the hybrid equilibrium coincides with the borrowing equilibrium. Setting $\bar{b}_H \equiv \bar{b}_B$, the hybrid equilibrium value function is therefore

$$V_H(b) = \begin{cases} \frac{\bar{C} - (\bar{C} + rb_H - y) \left(\frac{\bar{C} + rb_H - y}{\bar{C} + rb_H - y} \right)^{\frac{\rho}{r}}}{\rho} & \text{for } b \leq b_H \\ V_B(b) & \text{for } b \in (b_H, \bar{b}_H]. \end{cases} \quad (49)$$

The associated price schedule is

$$q_H(b) = \begin{cases} 1 & \text{for } b \leq b_H \\ q_B(b) & \text{for } b \in (b_H, \bar{b}_H]. \end{cases} \quad (50)$$

Finally, the policy function for consumption is

$$C_H(b) = \begin{cases} \bar{C} & \text{for } b < b_H \\ y - rb_H & \text{for } b = b_H \\ C_B & \text{for } b \in (b_H, \bar{b}_H]. \end{cases} \quad (51)$$

We state the following:

Proposition A.1. *Suppose neither the borrowing equilibrium nor the saving equilibrium exists. Specifically, suppose that $\underline{b}_S < \underline{b}_B$ and that there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b}$. Then a hybrid equilibrium exists.*

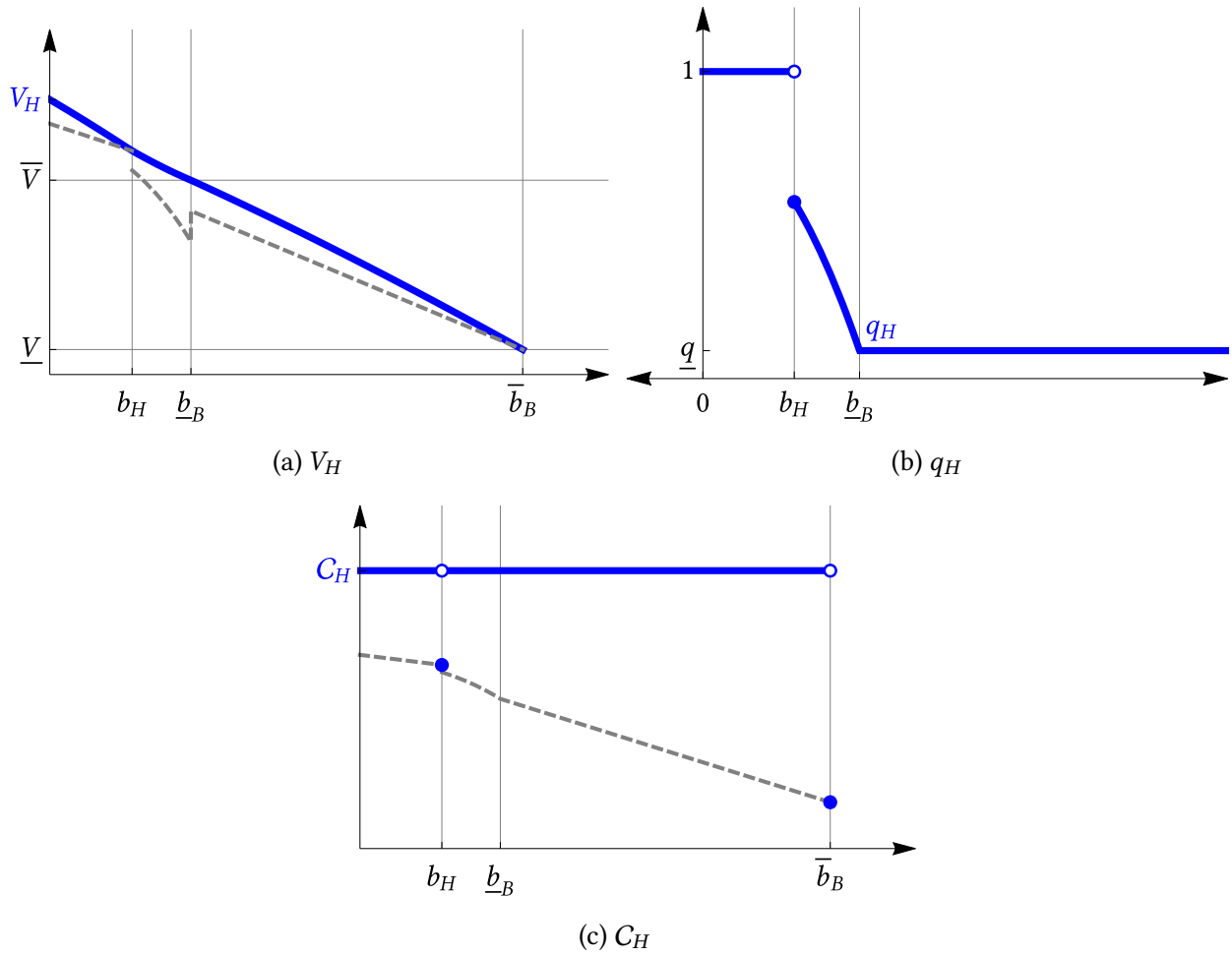
Proof. The conjectured price schedule q_H is consistent with the lenders’ break-even condition given the assumed government policy. Thus, to establish the conditions of an equilibrium, it is sufficient to prove that V_H is a solution to the government’s HJB.

- (i) For $b \in [\underline{b}_S, \bar{b}_B]$: By premise, $\underline{b}_B > \underline{b}_S$. This implies that $\rho V_B(\underline{b}_S) > \bar{V} = y - r\underline{b}_S \geq y - [r + \delta(1 - q_B(\underline{b}_S))]\underline{b}_S$. For $b > \underline{b}_S$, we have $\rho\bar{V} > y - rb$. As $V_B \geq \bar{V}$ for $b \leq \underline{b}_B$, we have $V_B(b) \geq y - rb$ for $b \in [\underline{b}_S, \underline{b}_B]$. From the proof of Proposition 5, this implies that $V_H(b) = V_B(b)$ satisfies the government's HJB on this domain. The proof of Proposition 5 extends this to $b \in [\underline{b}_B, \bar{b}_B]$ as well.
- (ii) For $b \leq \underline{b}_S$: Note that the premise implies there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b} \leq y - r\hat{b}$. The above established that $\rho V_B(b) > y - rb$ for $b \in [\underline{b}_S, \bar{b}_B]$. Hence, $\hat{b} < \underline{b}_S$. By continuity, there exists a $b_H \in (\hat{b}, \underline{b}_S)$ such that $\rho V_B(b_H) = y - rb_H$. Note as well that this implies $V'_B(b_H) = \lim_{b \downarrow b_H} V'_H(b) \geq -r/\rho$. From the expression for V_H , we have $\lim_{b \uparrow b_H} V_H(b_H) = -1 \leq -r/\rho$. Hence, V_H is either differentiable or has a convex kink at b_H , satisfying the conditions for a solution to the government's HJB at b_H . For $b < b_H$, $V'_H(b) \geq -1$, implying that the HJB is satisfied on this domain as well. Finally, $V_H(b) > \bar{V}$ for $b \leq b_H$, rationalizing the government's non-default on this domain.

□

This establishes that at least one of the three types of equilibria always exists. We note that the hybrid may coexist with the other equilibria as well. In fact, as $\bar{C} \rightarrow \infty$, the condition for multiplicity presented in Proposition 9 also implies the existence of a hybrid equilibrium.

Figure A.1: Hybrid Equilibrium



The figure depicts the value, price, and consumption functions in a hybrid equilibrium, respectively. The equilibrium functions are represented by the bold solid blue lines. The horizontal lines in Panel (a) represent the two default values. The dashed line in Panel (a) represents the stationary value function at the corresponding equilibrium prices. The dashed line in Panel (c) represents the level of consumption associated with the stationary value. The equilibrium is constructed with the same parameters as in Figure 1: $r = 1$, $\rho = 2$, $y = 1$, $\lambda = 2$, $\delta = 10$, $\bar{C} = 1.2$, $\underline{V} = .8y/\rho$, and $\bar{V} = .95y/\rho$. The value of \underline{C} is set low enough so that it never binds in equilibrium.

Appendix B: Proofs

This appendix contains all proofs except those for Propositions 1 and 4, which are presented in Appendix C, along with a discussion of viscosity solutions more generally.

B.1 Proof of Lemma 1

Proof. To generate a contradiction, suppose there is an efficient allocation $\{c, T\}$, with $T < \infty$. Note from (1) we have $V(T, c) = \underline{V}$. To see this, suppose instead that $V(T, c) > \underline{V}$; that is,

$$\begin{aligned} V(T, c) &= \sup_{T' \geq T} \int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s) ds + e^{-(\rho+\lambda)(T'-T)} \underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, c), \bar{V} \rangle ds \\ &> \underline{V}. \end{aligned}$$

Hence, there exists a $T' > T$ such that

$$\int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s) ds + e^{-(\rho+\lambda)(T'-T)} \underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, c), \bar{V} \rangle ds > \underline{V}.$$

This implies at time $t < T$,

$$\begin{aligned} &\int_t^T e^{-(\rho+\lambda)(s-t)} c(s) ds + e^{-(\rho+\lambda)(T-t)} \underline{V} + \lambda \int_t^T e^{-(\rho+\lambda)(s-t)} \max\langle V(s, c), \bar{V} \rangle ds < \\ &\int_t^{T'} e^{-(\rho+\lambda)(s-t)} c(s) ds + e^{-(\rho+\lambda)(T'-t)} \underline{V} + \lambda \int_t^{T'} e^{-(\rho+\lambda)(s-t)} \max\langle V(s, c), \bar{V} \rangle ds. \end{aligned}$$

Hence, T was never a sup of the original problem. This establishes that $V(T, c) = \underline{V}$.

Now consider an alternative allocation (\tilde{c}, ∞) . The alternative consumption allocation equals c for $t < T$, but differs for $t \geq T$. We choose $\tilde{c}(t) = (\rho + \lambda)\underline{V} - \lambda\bar{V} < y$ for $t \geq T$ so that for all $t \geq T$:

$$\begin{aligned} V(t, \tilde{c}) &= \frac{\tilde{c}(t) + \lambda\bar{V}}{\rho + \lambda} \\ &= \frac{(\rho + \lambda)\underline{V} - \lambda\bar{V} + \lambda\bar{V}}{\rho + \lambda} \\ &= \underline{V}. \end{aligned}$$

Thus, $V(0; c) = V(0; \tilde{c})$. Moreover, the alternative allocation delivers strictly more than zero to the lender in expectation for $t \geq T$ as $\tilde{c}(t) < y$. As the government is indifferent and the lender receives strictly more in expected present value, the original allocation is not efficient. \square

B.2 Proof of Lemma 2

Proof. Lemma 1 allows us to set $T = \infty$ in the planning problem (3) to obtain

$$P^*(v) = \sup_{c \in \mathcal{C}} \int_0^\infty e^{-\int_0^t r + \mathbb{1}_{[v(s) < \bar{V}]} \lambda ds} [y - c(t)] dt \quad (52)$$

subject to
$$\begin{cases} v(0) &= v \\ \dot{v}(t) &= -c(t) + \rho v(t) - \mathbb{1}_{[v(t) < \bar{V}]} \lambda [\bar{V} - v(t)], \end{cases}$$

defined on the domain $v \in \mathbb{V}$. P^* is bounded above by $(y - \underline{C})/r$ and below by $(y - \bar{C})/r$. To see that P^* is Lipschitz continuous in v , consider $v_1, v_2 \in \mathbb{V}$, with $v_2 > v_1$. A feasible strategy starting from $v(0) = v_2$ is to set consumption to \bar{C} until $v(t) = v_1$. Let Δ denote the time $v(t)$ reaches v_1 . Suppose $v(t) > \bar{V}$ for $t \in [0, \Delta_1)$ and $v(t) < \bar{V}$ for $t \in (\Delta_1, \Delta]$. Let $\Delta_2 = \Delta - \Delta_1$. If $v_2 < \bar{V}$, then $\Delta_1 = 0$ and if $v_1 > \bar{V}$, then $\Delta_2 = 0$. The dynamics of $v(t)$ imply

$$e^{-\rho \Delta_1} = \frac{\bar{C} - \rho \max\{v_2, \bar{V}\}}{\bar{C} - \rho \max\{v_1, \bar{V}\}}$$

$$e^{-(\rho + \lambda) \Delta_2} = \frac{\bar{C} + \lambda \bar{V} - (\rho + \lambda) \min\{v_2, \bar{V}\}}{\bar{C} + \lambda \bar{V} - (\rho + \lambda) \min\{v_1, \bar{V}\}}.$$

Using this, one can show that

$$1 - e^{-\rho \Delta_1 - (\rho + \lambda) \Delta_2} \leq L |v_2 - v_1|, \quad (53)$$

with $L \equiv (\rho + \lambda)/(\bar{C} - \rho V_{max}) \in (0, \infty)$.

As this is a feasible strategy for v_2 , integrating the objective function, we obtain

$$P^*(v_2) \geq (y - \bar{C}) \left(\frac{1 - e^{-r \Delta_1}}{r} + \frac{e^{-r \Delta_1} (1 - e^{-(r + \lambda) \Delta_2})}{r + \lambda} \right) + e^{-r \Delta_1 - (r + \lambda) \Delta_2} P^*(v_1).$$

As $y < \bar{C}$, we have

$$P^*(v_2) \geq (y - \bar{C}) \left(\frac{1 - e^{-r \Delta_1 - (r + \lambda) \Delta_2}}{r} \right) + e^{-r \Delta_1 - (r + \lambda) \Delta_2} P^*(v_1),$$

which implies

$$P^*(v_1) - P^*(v_2) \leq \left(\frac{\bar{C} - y}{r} + P^*(v_1) \right) \left(1 - e^{-r \Delta_1 - (r + \lambda) \Delta_2} \right).$$

As $P^*(v_1) \leq (y - \underline{C})/r$, we have

$$P^*(v_1) - P^*(v_2) \leq \left(\frac{\bar{C} - \underline{C}}{r} \right) \left(1 - e^{-r \Delta_1 - (r + \lambda) \Delta_2} \right).$$

As $\bar{C} > \underline{C}$ and $\rho \geq r$, this implies

$$\begin{aligned} P^*(v_1) - P^*(v_2) &\leq \left(\frac{\bar{C} - \underline{C}}{r} \right) \left(1 - e^{-\rho\Delta_1 - (\rho+\lambda)\Delta_2} \right) \\ &\leq \left(\frac{\bar{C} - \underline{C}}{r} \right) L |v_2 - v_1|, \end{aligned}$$

where the second line uses (53). As $v_1 < v_2$, and hence $P^*(v_1) \geq P^*(v_2)$ as P^* is the efficient frontier, we have

$$|P^*(v_1) - P^*(v_2)| \leq K |v_2 - v_1|,$$

where

$$K \equiv \left(\frac{\bar{C} - \underline{C}}{r} \right) L = \left(\frac{\bar{C} - \underline{C}}{\bar{C} - \rho V_{max}} \right) \left(\frac{\rho + \lambda}{r} \right).$$

Hence, P^* is Lipschitz continuous with coefficient $K \in (0, \infty)$. □

B.3 Proof of Proposition 2

Proof. We need to check the conditions of Proposition 1. Note that P_B^* is bounded, Lipschitz continuous, and differentiable everywhere except \bar{V} , where $\lim_{v \uparrow \bar{V}} P_B^*(v) < \lim_{v \downarrow \bar{V}} P_B^*(v)$. This inequality implies that condition (ii) in the proposition is irrelevant. Condition (iii) of Proposition 1 is satisfied trivially. Condition (iv) is satisfied by construction.

At points of differentiability, the first-order condition for consumption requires $P_B^*(v) \leq -1$ for $c = \bar{C}$ to be optimal. Starting with $v \in [\underline{V}, \bar{V})$, differentiating the candidate function yields $P_B^*(v) \leq -1$. Hence \bar{C} is optimal, and P_B^* satisfies the HJB on this domain. Turning to $v > \bar{V}$, note that $P_B^*(v)$ is concave on this domain. Thus, if $\lim_{v \downarrow \bar{V}} P_B^*(v) \leq -1$, then $P_B^*(v) \leq -1$ for $v \in (\bar{V}, V_{max}]$. We have

$$\lim_{v \downarrow \bar{V}} P_B^*(v) = -\frac{\bar{C} - y + rP_B^*(\bar{V})}{\bar{C} - \rho\bar{V}}.$$

This quantity is less than -1 when $rP_B^*(\bar{V}) \geq y - \rho\bar{V}$. This is the condition stated in the proposition. This condition is necessary and sufficient for P_B^* to satisfy the HJB on (\bar{V}, V_{max}) . Moreover, it is sufficient to ensure that condition (v) of Proposition 1 is satisfied. □

B.4 Proof of Proposition 3

Proof. The proposed solution P_S^* is differentiable everywhere save \bar{V} and v^I . At \bar{V} we have $\lim_{v \uparrow \bar{V}} P_S^*(v) \geq -1 \geq \lim_{v \downarrow \bar{V}} P_S^*(v)$. Hence, condition (ii) of Proposition 1 is relevant and is satisfied by the candidate value function. P_S^* satisfies condition (iii) at v^I as it features a convex kink by construction. Condition (iv) is also satisfied by construction.

On the domain $v \in (\bar{V}, V_{max}]$, we have $P_S^*(v) \leq -1$, and hence P_S^* satisfies the HJB as well as condition (v) of Proposition 1.

Turning to $v < \bar{V}$, we now show that $P_S^*(\bar{V}) \geq P_B^*(\bar{V})$ is necessary and sufficient for P_S^* to satisfy the conditions of Proposition 1.

For sufficiency, suppose that $P_S^*(\bar{V}) \geq P_B^*(\bar{V})$. Let $X \equiv \{v \in [\underline{V}, \bar{V}] | P_S^*(v) \geq P_B^*(v)\} = [\max\{v^I, \underline{V}\}, \bar{V}]$. On the domain X , $P_S^*(v) = \hat{P}(v)$. One can show that $\hat{P}'(v) \geq -1$ if and only if $\hat{P}(v) \geq (y - (\rho + \lambda)v + \lambda\bar{V})/(r + \lambda)$. As the latter term is the value associated with setting $\dot{v} = 0$, the inequality is satisfied as $\hat{P}(v) \geq P_B^*(v) \geq (y - (\rho + \lambda)v + \lambda\bar{V})/(r + \lambda)$. Hence $c = \underline{C}$ is optimal on X , and the HJB is satisfied. If $\hat{P}(\underline{V}) \geq P_B^*(\underline{V})$, then $X = [\underline{V}, \bar{V}]$, and hence the HJB is satisfied on the whole domain \mathbb{V} . If instead there exists $v^I > \underline{V}$, then the HJB is satisfied for $v < v^I$ from Proposition 2.

For necessity, suppose instead that $P_S^*(\bar{V}) < P_B^*(\bar{V})$. Comparison of the slopes implies that as long as $P_S^*(v) < P_B^*(v)$ for $v \in [\underline{V}, \bar{V}]$, then $P_S^{*\prime}(v) < P_B^{*\prime}(v)$, and the two lines will never cross. Moreover, $P_B^{*\prime}(v) \leq -1$, and hence $P_S^{*\prime}(v) < -1$. This implies that $c = \underline{C}$ is strictly sub-optimal and the HJB is violated. \square

B.5 Proof of Lemma A.1

Lemma A.1. *In any competitive equilibrium such that $q(b) \in [\underline{q}, 1]$ for $b \in \mathbf{B} = [-\bar{a}, \bar{b}]$, V is bounded, strictly decreasing, and Lipschitz continuous on \mathbf{B} .*

Proof. The boundedness of V follows directly from $\bar{C}/\rho \geq V(b) \geq \underline{V}$ for any $b \in \mathbf{B}$.

To see that V is strictly decreasing, suppose $b_1 > b_2$ for $b_1, b_2 \in \mathbf{B}$. If $b_2 = -\bar{a} \equiv (y - \bar{C})/\rho$, then $V(b_2) = \bar{C}/\rho > V(b_1)$, where the latter inequality follows from the budget set at $b_1 > b_2$. Now consider the following policy starting from $b_2 \in (-\bar{a}, b_1)$: Set $c = \bar{C}$ until $b(t) = b_1$. As

$$\dot{b}(t) = \frac{c + (r + \delta)b(t) - y}{q(b(t))} - \delta b,$$

and $\bar{C} > y - rb \geq y - [r + \delta(1 - q(b))]b$ for $b \geq b_2$, we have $\dot{b}(t) > 0$. Let $\tilde{t} \in (0, \infty)$ denote when $b(t) = b_1$. As it is feasible for the government to follow this policy and not default while doing so, we have

$$V(b_2) \geq \int_0^{\tilde{t}} \bar{C} dt + e^{-\rho\tilde{t}} V(b_1) = (1 - e^{-\rho\tilde{t}}) \frac{\bar{C}}{\rho} + e^{-\rho\tilde{t}} V(b_1).$$

Subtracting $V(b_1)$ from both sides yields:

$$V(b_2) - V(b_1) \geq (1 - e^{-\rho\tilde{t}}) \left(\frac{\bar{C}}{\rho} - V(b_1) \right) > 0.$$

For continuity, we proceed in a similar fashion. Starting from b_1 , consider the policy of setting $c = \underline{C}$ until $b(t) = b_2$. Let t^* denote the time where $b(t) = b_2$. Given that $\underline{C} < y - (r + \delta)\bar{b} \leq y - (r + \delta)b(t)$ and $q(b(t)) \in [\underline{q}, 1]$, $t^* < \infty$. Moreover, the same statements imply that

$$b_2 - b_1 \geq \int_0^{t^*} (\underline{C} + rb(t) - y) dt \geq \int_0^{t^*} (\underline{C} + r\bar{b} - y) dt = (\underline{C} + r\bar{b} - y) t^*,$$

where the first inequality follows from $q(b) \leq 1$.

The above implies that $t^* \geq L|b_1 - b_2|$, with $L \equiv (y - r\bar{b} - \underline{C})^{-1} \in (0, \infty)$.

As this is a feasible strategy, we have

$$V(b_1) \geq \int_0^{t^*} e^{-\rho t} \underline{C} dt + e^{-\rho t^*} V(b_2) = (1 - e^{-\rho t^*}) \frac{\underline{C}}{\rho} + e^{-\rho t^*} V(b_2),$$

where the inequality in the first line also reflects that the right-hand side is the value assuming the government never defaults, which is weakly below the optimal default policy. Subtracting $V(b_2)$ from both sides and rearranging, we have

$$V(b_2) - V(b_1) \leq (1 - e^{-\rho t^*}) \left(V(b_2) - \frac{C}{\rho} \right).$$

Using the fact that $\bar{C}/\rho > V(b) \geq \underline{V} > \underline{C}/\rho$ and $1 - e^{-\rho t^*} \leq t^*$, we have

$$0 < V(b_2) - V(b_1) \leq t^* \left(V(b_2) - \frac{C}{\rho} \right) \leq L \left(\frac{\bar{C} - \underline{C}}{\rho} \right) |b_1 - b_2|.$$

Hence, $|V(b_2) - V(b_1)| \leq K|b_2 - b_1|$ with $K \equiv L \left(\frac{\bar{C} - \underline{C}}{\rho} \right) / \rho \in (0, \infty)$. \square

B.6 Proof of Proposition 5

Proof. By construction, the price schedule q_B is consistent with the lenders' break-even condition, given the conjectured government policy. The remaining step is to verify if and when the government's policy is optimal given the conjectured q_B . Hence, to prove the proposition, we need to establish that V_B satisfies the conditions of Proposition 4 if and only if (32) holds.

For \bar{C} to be optimal for all $b < \bar{b}_B$, the first-order condition for the HJB requires $1 + V'_B(b)/q_B(b) \geq 0$ wherever $V'_B(b)$ exists. Thus, if $V'_B(b) \geq -q_B(b)$, then $c = \bar{C}$ is optimal. Recalling that V_B was constructed by assuming that the Hamiltonian is maximized at $c = \bar{C}$, then $V'_B(b) \geq -q_B(b)$ is both necessary and sufficient to verify that the HJB is satisfied at points of differentiability.

We proceed to show that (32) is equivalent to $V'_B(b) \geq -q_B(b)$ at points of differentiability.

For $b < 0$, we have

$$\rho V_B(b) = \bar{C} - (\bar{C} - \rho V_B(0)) \left(\frac{\bar{C} + rb - y}{\bar{C} - y} \right)^{\frac{\rho}{r}}.$$

Note that V_B is concave on this domain. For \bar{C} to be optimal, it is therefore sufficient that $\lim_{b \uparrow 0} V'_B(b) \geq -1$. This will be true if and only if $\rho V_B(0) \geq y$. Hence, the condition in equation (32) evaluated at $b = 0$ is necessary and sufficient for the HJB to hold for $b \in (-\bar{a}, 0)$. For $b = -\bar{a} = (y - \bar{C})/\rho$, we have $V_B(-\bar{a}) = \bar{C}/\rho$, which is condition (iv) in Proposition 4.

For $b \in (0, \underline{b}_B]$, we use the fact that $V_B(b) = V_B^*(q_B(b)b)$, where V_B^* is the inverse of P_B^* . Hence, $V'_B(b) = V_B^{*\prime}(q_B(b)b) (q_B(b) + q'_B(b)b)$. From the lenders' break-even condition, in the Safe Zone, we have $(r + \delta)q_B(b) = q'_B(b)b = q'_B(b) \left(\bar{C} + [r + \delta(1 - q_B(b))]b - y \right)$. Substituting in and rearranging, we have for $b \in (0, \underline{b}_B]$

$$V'_B(b) = -q_B(b) \left(\frac{\bar{C} - \rho V_B(b)}{\bar{C} - [r + \delta(1 - q_B(b))]b - y} \right).$$

Hence, for $b \in (0, \underline{b}_B]$, the HJB is satisfied if and only if $\rho V_B(b) \geq y - [r + \delta(1 - q_B(b))]b$, which is the condition in equation (32).

For $b \in (\underline{b}_B, \bar{b}_B]$, we have $q_B(b) = \underline{q}$ and

$$(\rho + \lambda)V_B(b) = \bar{C} + \lambda\bar{V} - \left(\bar{C} + \lambda\bar{V} - (\rho + \lambda)\underline{V}\right) \left(\frac{\bar{C} - y + (r + \lambda)\underline{q}b}{\bar{C} - y + (r + \lambda)\underline{q}\bar{b}_B}\right)^{\frac{\rho + \lambda}{r + \lambda}}.$$

Note that $V_B(b)$ is concave in b , hence we need to check the condition at $b \rightarrow \bar{b}_B$. We have for $b \in (\underline{b}_B, \bar{b}_B]$

$$V_B'(b) \geq -\frac{\bar{C} + \lambda\bar{V} - (\rho + \lambda)\underline{V}}{\bar{C} - y + (r + \lambda)\underline{q}\bar{b}_B} = -1,$$

where the final equality uses the definition of \bar{b}_B ; hence, for this region the optimality condition always holds.

By construction, for $b = \bar{b}_B$, condition (v) of Proposition 1 is satisfied.

Note that as $V_B(\underline{b}_B) = \bar{V}$, the derivative of V_B is continuous at \underline{b}_B . The only point of non-differentiability is $b = 0$. In particular, note that $\lim_{b \downarrow 0} V_B'(b) = -\lim_{b \downarrow 0} q_B(b)(\bar{C} - \rho V_B(0))(\bar{C} - y)$. Hence, if $\lim_{b \downarrow 0} q_B(b) < 1$, then there is a convex kink at $b = 0$. This is consistent with condition (iii) in Proposition 4.

Hence, the conditions of Proposition 4 hold if and only if (32) holds. \square

B.7 Proof of Proposition 6

Proof. There are three claims in the proposition:

Part (i). If a borrowing allocation is efficient, it must be that

$$rP_B^*(v) \geq y - \rho V$$

for any $V \geq \bar{V}$ (from footnote 14). This implies that

$$V_B^*(P) \geq \frac{y - rP}{\rho}$$

for $P \leq P_B(\bar{V})$. Using that $P = q_B(b)b$ and that $V_B(b) = V_B^*(q_B(b)b)$, we have that

$$\begin{aligned} V_B(b) &= V_B^*(q_B(b)b) \geq \frac{y - rq_B(b)b}{\rho} \\ &= \frac{y - \frac{rq_B(b)}{r + \delta(1 - q_B(b))}(r + \delta(1 - q_B(b)))b}{\rho} \\ &\geq \frac{y - (r + \delta(1 - q_B(b)))b}{\rho} \text{ for all } b \in [0, \underline{b}_B], \end{aligned}$$

where the last inequality follows from $rq_B(b) \leq r + \delta(1 - q_B(b))$ for all $\delta \geq 0$, $q_B(b) \leq 1$ and $b \geq 0$. And thus condition (7) is satisfied.

Part (ii). Let $p = q_B(b)b$. For $b \in [0, \underline{b}_B]$, Condition (7) becomes

$$\rho V_B^*(p) - (y - (r + \delta)b + \delta p) \geq 0$$

Now, from the price equation (31), we have

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}} \right)^{\frac{r}{r+\delta}} = \frac{\bar{C} - y + rp}{\bar{C} - y + r\underline{p}}, \quad (54)$$

where $\underline{p} \equiv P_B^*(\bar{V}) = \underline{q}\underline{b}_B$. From this expression, we can define $q_B(b) = F(\delta, p)$, holding the other parameters constant. Recall that condition (7) is restricted to $b \in [0, \underline{b}_B]$; hence, the domain of interest for p is $[0, \underline{p}]$, which is independent of δ . We shall use the fact that

$$\frac{\partial F(\delta, p)}{\partial \delta} = \frac{1 - F(\delta, p)}{r + \delta} \left(\underline{q} + \ln \left(\frac{1 - \underline{q}}{1 - F(\delta, p)} \right) \right), \quad (55)$$

keeping in mind that $\underline{q} = (r + \delta)/(r + \delta + \lambda)$ and hence varies with δ .

Condition (7) can be written:

$$G(\delta, p) \equiv \rho V_B^*(p) - y + (r + \delta)p/F(\delta, p) - \delta p.$$

Taking the derivative with respect to δ , we have that:

$$\begin{aligned} \frac{\partial G(\delta, p)}{\partial \delta} &= \frac{p}{F(\delta, p)} - p - \frac{(r + \delta)p}{F(\delta, p)^2} \frac{\partial F(\delta, p)}{\partial \delta} \\ &= \frac{p}{F(\delta, p)} \left(1 - F(\delta, p) - \frac{(r + \delta)}{F(\delta, p)} \frac{\partial F(\delta, p)}{\partial \delta} \right) \\ &= \frac{p(1 - F(\delta, p))}{F(\delta, p)^2} \left(F(\delta, p) - \frac{(r + \delta)}{(1 - F(\delta, p))} \frac{\partial F(\delta, p)}{\partial \delta} \right) \\ &= \frac{p(1 - F(\delta, p))}{F(\delta, p)^2} \left(F(\delta, p) - \underline{q} - \ln \left(\frac{1 - \underline{q}}{1 - F(\delta, p)} \right) \right). \end{aligned}$$

Note that $\partial G/\partial \delta \leq 0$ if

$$F(\delta, p) - \underline{q} - \ln \left(\frac{1 - \underline{q}}{1 - F(\delta, p)} \right) \leq 0.$$

For $p = \underline{p}$, $F(\delta, p) = \underline{q}$, and this term is zero. Moreover, this expression is increasing in p as $\partial F/\partial p < 0$. Hence, $\partial G(\delta, p)/\partial \delta \leq 0$ for $p \in [0, \underline{p}]$. Thus, if $G(\delta_0, p) \geq 0$, then $G(\delta, p) \geq 0$ for $\delta \in [0, \delta_0]$.

Part (iii). The fact that saving is efficient implies

$$\frac{y - \rho \bar{V}}{r} > P_B^*(\bar{V}) = \underline{q}\underline{b}_B,$$

where the last equality follows from the definition of \underline{b}_B . By continuity, there exists a $V_0 > \bar{V}$ such that

$$\frac{y - \rho V_0}{r} > P_B^*(V_0) \equiv p_0 < \underline{p},$$

where the last inequality follows from the fact that P_B^* is strictly decreasing. As $V_0 = V_B^*(p_0)$ by definition, this is equivalent to

$$\rho V_B^*(p_0) < y - rp_0.$$

Evaluated at $p = p_0$, condition (7) is

$$G(\delta, p_0) = \rho V_B^*(p_0) - y + \frac{rp_0}{F(\delta, p_0)} + p_0 \delta \left(\frac{1}{F(\delta, p_0)} - 1 \right). \quad (56)$$

Note that $\lim_{\delta \rightarrow \infty} \underline{q} = 1$, and hence $F(\delta, p) \geq \underline{q}$ also converges to 1. Hence, $\rho V_B^*(p_0) - y + rp_0/F(\delta, p_0) \rightarrow \rho V_B^*(p_0) - y + rp_0 < 0$. We now show that the last term in (56) converges to zero; that is, $\delta(1 - F(\delta, p_0)) \rightarrow 0$. From the definition of F in (54), we have

$$\delta(1 - F(\delta, p_0)) = \frac{\lambda \delta}{r + \delta + \lambda} \left(\frac{\bar{C} - y + rp_0}{\underbrace{\bar{C} - y + rp}_{< 1}} \right)^{\frac{r + \delta}{r}}.$$

As the ratio raised to the power $(r + \delta)/r$ is strictly less than one as $p_0 < \underline{p}$, the right-hand side goes to zero as $\delta \rightarrow \infty$. Hence, there exists a δ_1 such that for all $\delta > \delta_1$, $G(\delta, p_0) < 0$, violating the condition for the borrowing equilibrium. \square

B.8 Proof of Proposition 7

Proof. We proceed to show the necessity and sufficiency parts of the proposition independently.

The “only if” part. Toward a contradiction, suppose that $V_S(\underline{b}_S) < V_B(\underline{b}_S)$ (or equivalently $\underline{b}_B > \underline{b}_S$), and the conjectured saving equilibrium is indeed an equilibrium. First, note that $q_S(b) \in [\underline{q}, 1]$, as the government defaults only with the arrival of $V^D = \bar{V}$.

By construction, $\hat{V}(\underline{b}_S) = V_S(\underline{b}_S) = \bar{V}$. As V_S is strictly decreasing, we have $V_S(\underline{b}_B) < \bar{V} = V_B(\underline{b}_B)$. Hence, V_S and V_B do not intersect in $[\underline{b}_S, \underline{b}_B]$, and $b^I > \underline{b}_S$, and $V_S(\underline{b}_B) = \hat{V}(\underline{b}_B)$.

We also have for $b \geq \underline{b}_B$: $\hat{V}'(b) = -q_S(b) \leq -\underline{q} \leq V_B'(b)$, where the latter inequality uses a property of the borrowing allocation value function (shown in the proof of Proposition 5). This implies that $\hat{V}(b) < V_B(b)$ for all $b \geq \underline{b}_B$, and there is no point of intersection to generate b^I and $V_S = \hat{V}$ for all $b \in [\underline{b}_S, \bar{b}_S]$. Now at \bar{b}_S , we must have (as an equilibrium requirement) that $\hat{V}(\bar{b}_S) = \underline{V} < V_B(\bar{b}_S)$, where the latter inequality follows from the fact that $\hat{V} < V_B$ on this domain. Thus, $\bar{b}_S < \bar{b}_B$. However, we have

$$\begin{aligned} (\rho + \lambda)\underline{V} &= y - [r + \delta(1 - \underline{q})]\bar{b}_B + \lambda\bar{V} \\ &< y - [r + \delta(1 - \underline{q})]\bar{b}_S + \lambda\bar{V} \\ &\leq y - [r + \delta(1 - q_S(\bar{b}_S))]\bar{b}_S + \lambda\bar{V} \\ &= (\rho + \lambda)\hat{V}(\bar{b}_S) = (\rho + \lambda)\underline{V}, \end{aligned}$$

where the first line uses the definition of \bar{b}_B ; the second line uses $\bar{b}_B > \bar{b}_S$; the third uses $q_S(b) \geq \underline{q}$; and

the final two equalities use the fact that $\hat{V}(b)$ is the stationary value at price q_S and the definition of \bar{b}_S , respectively. Hence, we have generated a contradiction.

The “if” part. We first verify that V_S satisfies the conditions of Proposition 4 and establish that q_S is a valid equilibrium price schedule.

First, consider the government’s problem.

Condition (iv) and (v) of Proposition 4 are satisfied by construction. For $b = \underline{b}_S$, condition (ii) of Proposition 4 applies and is satisfied by construction.

For $b < \underline{b}_S$, the conjectured value function is differentiable. For the HJB to hold with $c = \bar{C}$ given that $q_S(b) = 1$ in this region, we require $V'_S(b) \geq -q_S(b) = -1$. On this domain, $V_S(b) = V_S^*(b)$, where the latter is the inverse of the efficient solution P_S^* . As $P_S^{*\prime}(v) \leq -1$, we have $V'_S(b) \geq -1 = -q_S(b)$. Hence, $c = \bar{C}$ is indeed optimal, and the HJB holds with equality.

For $b \in (\underline{b}_S, b^I)$, $V_S(b) = \hat{V}(b)$, and thus V_S is differentiable and satisfies the HJB with equality by construction. Note that if $q_S(b) \geq \bar{q}$ (something we check below), then $\dot{b} \leq 0$ in (b_S, b^I) by equation (19) (consistent with the equilibrium conjecture that the government is saving in this region). This implies that $C_S(b) = \hat{C}(b) \leq \bar{C}$, and thus the conjectured policy function is a valid one (recall that we are assuming that \underline{C} is sufficiently low and can thus be ignored as a constraint).

For $b \in (b^I, \bar{b}_S)$, $V_S(b) = V_B(b)$ and differentiability of V_B implies that V_S is differentiable. The proof of Proposition 5 establishes that the HJB holds with equality in this domain, given that $q^S(b) = \underline{q}$.

This confirms that condition (i) of Proposition 4 holds.

If $b^I \in (\underline{b}_S, \bar{b}_S)$, $V_S(b^I) = V_B(b^I)$, and there is a potential point of non-differentiability at b^I . If $q_S(b^I) \geq \underline{q}$ (something that we check below), we have that this kink is convex. Thus, condition (iii) of Proposition 4 holds.

Hence, given the conjectured q_S , the value function is a viscosity solution to the government’s HJB equation.

Next, let us consider the price function. The only thing left to check is that $q^S(b) \in [\underline{q}, 1]$ for $b \in (\underline{b}_S, b^I)$, where $b^I \in (\underline{b}_B, \bar{b}_B)$. In this region, $q^S(b) \leq 1$ by equation (39). In addition,

$$\begin{aligned} (\rho + \lambda)V_S(b) &= y - [r + \delta(1 - q_S(b))]b + \lambda\bar{V} \\ &\geq (\rho + \lambda)V_B(b) \\ &\geq y - [r + \delta(1 - \underline{q})]b + \lambda\bar{V}, \end{aligned}$$

where the first equality and second inequality follow from the equilibrium construction on $b \in (\underline{b}_S, b^I)$. The last inequality follows from the construction of $V_B(b)$ for $b \in (\underline{b}_B, \bar{b}_B)$. Comparison of the first and last lines establishes that $q_S(b) \geq \underline{q}$. \square

B.9 Proof of Proposition 8

Proof. The fact that the efficiency of saving is a necessary condition for a saving equilibrium is established in the text. Turning to equation (44), multiply both sides of equation (43) by \underline{q} to obtain the following necessary and sufficient condition:

$$\underline{q}P_S^*(\bar{V}) \geq \underline{q}\underline{b}_B = P_B^*(\bar{V}).$$

Using $\underline{q} = (r + \delta)/(r + \delta + \lambda)$ and the fact that $P_S^*(\bar{V}) > P_B^*(\bar{V})$, we solve for δ to obtain

$$\delta \geq \frac{\lambda P_B^*(\bar{V})}{P_S^*(\bar{V}) - P_B^*(\bar{V})} - r = \frac{(r + \lambda)P_B^*(\bar{V}) - rP_S^*(\bar{V})}{P_S^*(\bar{V}) - P_B^*(\bar{V})} \geq 0,$$

where the last inequality is strict when $\rho > r$, as seen in the definition of P_B^* . Thus, this is a necessary and sufficient condition for the saving equilibrium, proving the proposition. \square

B.10 Proof of Proposition 9

Proof. Note that $P_B^*(v)$ is increasing in \bar{C} . Hence,

$$\begin{aligned} P_B^*(\bar{V}) &\leq \lim_{\bar{C} \rightarrow \infty} P_B^*(\bar{V}) \\ &= \frac{y - \rho\bar{V} + (\rho - r)(\bar{V} - \underline{V})}{r + \lambda} \\ &= \frac{rP_S^*(\bar{V}) + (\rho - r)(\bar{V} - \underline{V})}{r + \lambda}. \end{aligned}$$

Then a sufficient condition for saving to be strictly efficient is

$$P_S^*(\bar{V}) > \frac{rP_S^*(\bar{V}) + (\rho - r)(\bar{V} - \underline{V})}{r + \lambda},$$

or

$$\frac{\lambda}{\rho - r} > \frac{r(\bar{V} - \underline{V})}{y - \rho\bar{V}},$$

which is the last inequality in the proposition.

Similarly,

$$P_S^*(\bar{V}) - P_B^*(\bar{V}) \geq \frac{\lambda P_S^*(\bar{V}) - (\rho - r)(\bar{V} - \underline{V})}{r + \lambda},$$

and

$$\begin{aligned} \frac{\lambda P_B^*(\bar{V})}{P_S^*(\bar{V}) - P_B^*(\bar{V})} - r &\leq \frac{\lambda \left(rP_S^*(\bar{V}) + (\rho - r)(\bar{V} - \underline{V}) \right)}{\lambda P_S^*(\bar{V}) - (\rho - r)(\bar{V} - \underline{V})} - r \\ &= \frac{(r + \lambda)(\rho - r)(\bar{V} - \underline{V})}{\lambda P_S^*(\bar{V}) - (\rho - r)(\bar{V} - \underline{V})} \equiv \underline{\delta}. \end{aligned}$$

From Proposition 8, a sufficient condition for a saving equilibrium, given that saving is efficient, is that δ is greater than $\underline{\delta}$. Note that $\underline{\delta}$ is strictly positive and independent of \bar{C} .

For the borrowing equilibrium, we need to show that the condition in equation (7) is satisfied as \bar{C} becomes arbitrarily large. Specifically, fix any $\delta = \bar{\delta} > \underline{\delta}$. Define

$$A(b) \equiv \rho V_B(b) - (y - [r + \delta(1 - q_B(b))]b),$$

where A implicitly depends on \bar{C} and δ . Note by condition (7) in Proposition (5) that if $A(b) > 0$ on $[0, \bar{b}_B] \supseteq [0, \underline{b}_B]$, then a borrowing equilibrium exists.

To establish the properties of $A(b)$ as $\bar{C} \rightarrow \infty$, first note that \bar{b}_B is independent of \bar{C} . In addition,

$$\lim_{\bar{C} \rightarrow \infty} V_B(b) = \underline{V} + \underline{q}(\bar{b}_B - b),$$

where we use the fact that $q_B(b) \rightarrow \underline{q}$ for $b \in [0, \bar{b}_B]$ as $\bar{C} \rightarrow \infty$. As the point-wise limit is continuous in b , and by Lemma A.1 $V_B(b)$ is monotonic given \bar{C} , the convergence is uniform on the compact set $[0, \bar{b}_B]$ (see Theorem A of Buchanan and Hildebrandt (1908)).

Similarly, for $b \in [0, \bar{b}_B]$,

$$\lim_{\bar{C} \rightarrow \infty} \{y - [r + \delta(1 - q_B(b))]b\} = y - [r + \delta(1 - \underline{q})]b = y - (r + \lambda)\underline{q}b.$$

Again, the convergence is uniform by the same logic.

Hence, $A(b)$ converges uniformly on $[0, \bar{b}_B]$ to

$$\bar{A}(b) \equiv \lim_{\bar{C} \rightarrow \infty} A(b) = \rho\underline{V} + \rho\underline{q}(\bar{b}_B - b) - (y - (r + \lambda)\underline{q}b).$$

We now establish that $\bar{A}(b) > 0$ for $b \in [0, \bar{b}_B]$. The linearity of $\bar{A}(b)$ in b implies that if the inequality holds for $b = 0$ and $b = \bar{b}_B$, it is satisfied for all intermediate points. For $b = 0$, we have

$$\begin{aligned} \bar{A}(0) &= \rho\underline{V} + \rho\underline{q}\bar{b}_B - y \\ &= \frac{(\rho - r - \lambda)(y - \rho\underline{V}) + \rho\lambda(\bar{V} - \underline{V})}{r + \lambda} \\ &= \frac{(\rho - r - \lambda)(y - \rho\bar{V}) + (\rho - r)\rho(\bar{V} - \underline{V})}{r + \lambda} > 0, \end{aligned}$$

where the second line uses the definition of $\underline{q}\bar{b}_B$ and the final inequality uses the condition in the proposition. Similarly,

$$\begin{aligned} \bar{A}(\bar{b}_B) &= \rho\underline{V} - (y - (r + \lambda)\underline{q}\bar{b}) \\ &= \lambda(\bar{V} - \underline{V}) > 0. \end{aligned}$$

Hence, $\min_{b \in [0, \bar{b}_B]} \bar{A}(b) = \min\{\bar{A}(0), \bar{A}(\bar{b})\} > 0$.

As $A \rightarrow \bar{A}$ uniformly on $[0, \bar{b}_B]$, for every $\epsilon > 0$, there exists an M such that for all $\bar{C} > M$, we have $A(b) > \bar{A}(b) - \epsilon$ for $b \in [0, \bar{b}_B]$. Setting $\epsilon < \min_{b \in [0, \bar{b}_B]} \bar{A}(b)$, we have $A(b) > 0$ for all $b \in [0, \bar{b}_B]$ and $\bar{C} > M$. By Proposition 5, this is sufficient to establish the existence of a borrowing equilibrium for $\delta = \bar{\delta}$ when $\bar{C} > M$. By part (ii) of Proposition 6, we have a borrowing equilibrium for all $\delta \in [0, \bar{\delta}]$.

Combining results, there exists a non-empty interval $\Delta \equiv [\underline{\delta}, \bar{\delta}]$ and M such that for all $\bar{C} > M$ and $\delta \in \Delta$, both saving and borrowing equilibria coexist. \square

B.11 Proof of Proposition 10

Proof. We first sketch out the borrowing equilibrium under the assumed policy. Let $\{V_B^P, q_B^P\}$ denote the equilibrium policy and price functions. The conjectured policy is for the government to borrow to \bar{b}_B ,

which is the endogenous limit in the borrowing equilibrium absent the policy. From (47), it is optimal for the bondholders to sell their bonds at price $q^* > \underline{q}$ as soon as $b = \underline{b}_B^P$, where the latter is defined by $V^P(\underline{b}_B^P) = \bar{V}$. That is, bondholders sell their bonds to the third party as soon as debt enters the Crisis Zone. We have

$$\begin{aligned} V_B^P(\bar{b}_B) &= \frac{y - [r + \delta(1 - q^*)]\bar{b}_B + \lambda\bar{V}}{\rho + \lambda} \\ &= \underline{V} + \frac{\delta(q^* - \underline{q})\bar{b}_B}{\rho + \lambda}. \end{aligned}$$

The last term reflects that the government rolls over debt at q^* rather than \underline{q} once it reaches the borrowing limit. The expression assumes that the government defaults upon the arrival of \bar{V} . To see that this is optimal, note that the alternative of never defaulting yields the value

$$\frac{y - [r + \delta(1 - q^*)]\bar{b}_B}{\rho} \leq \frac{y - r\bar{b}_B}{\rho} < \frac{y - r\underline{b}_S}{\rho} = \bar{V}.$$

Facing $q_B^P(b) = q^*$ in the Crisis Zone, the government's value can be obtained from the HJB, and it is straightforward to verify that the first-order condition for $c = \bar{C}$ holds on this domain. As $q^* > \underline{q}$, $\underline{b}_B^P > \underline{b}_B$, where the latter is the benchmark borrowing equilibrium's threshold for the Safe Zone. Note as well that $q^* > \underline{q}$ implies that the third party takes a loss in expectation in the Crisis Zone.

For $b \in [0, \underline{b}_B^P]$, bondholders purchase debt at price $q_B^P(b)$ and collect r plus maturing principal until $b = \underline{b}_B^P$, at which point they sell at q^* . The equilibrium is recovered by solving the system of differential equations:

$$\begin{aligned} \rho V_B^P(b) &= \bar{C} + V_B^{P'}(b)\dot{b} \\ (r + \delta)q_B^P(b) &= r + \delta + q_B^{P'}(b)\dot{b} \\ \dot{b} &= \frac{\bar{C} + (r + \delta)b - y}{q_B^P(b)} - \delta b, \end{aligned}$$

with the boundary conditions $V_B^P(\underline{b}_B^P) = \bar{V}$ and $q_B^P(\underline{b}_B^P) = q^*$. Note that these equations are identical to those in the benchmark borrowing equilibrium except that the boundary condition $\underline{b}_B^P > \underline{b}_B$ and $q^* > \underline{q}$.

As is the case in the benchmark equilibrium, a necessary and sufficient condition for V_B^P to be a solution to the government's problem when facing q_B^P is

$$V_B^P(b) \geq \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho},$$

for all $b \in [0, \underline{b}_B^P]$. Following the same approach as in the proof of Proposition 9, we show that this inequality holds as $\bar{C} \rightarrow \infty$ uniformly over the full debt domain $[0, \bar{b}_B]$.

As $\bar{C} \rightarrow \infty$, we have for $b \in [0, \bar{b}_B]$,

$$\begin{aligned} \lim_{\bar{C} \rightarrow \infty} V_B^P(b) &= V_B^P(\bar{b}_B) + q^*(\bar{b}_B - b) \\ \lim_{\bar{C} \rightarrow \infty} \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} &= \frac{y - [r + \delta(1 - q^*)]b}{\rho}. \end{aligned}$$

Recall from the proof of Proposition 9, that

$$\bar{A}(b) = \underline{V} + \underline{q}(\bar{b}_B - b) - \frac{y - [r + \delta(1 - \underline{q})]b}{\rho} \geq 0$$

under the conditions of the proposition. Note that this implies

$$\lim_{\bar{C} \rightarrow \infty} \left(V_B^P(b) - \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \right) = \bar{A}(b) + \frac{\delta(q^* - \underline{q})\bar{b}_B}{\rho + \lambda} + (q^* - \underline{q})(\bar{b}_B - b) - \frac{\delta(q^* - \underline{q})}{\rho}b.$$

This expression is linear in b , and hence it is sufficient to verify the inequality at the endpoints $b = 0$ and $b = \bar{b}_B$. The fact that $\bar{A}(0) > 0$ and $q^* > \underline{q}$ implies that the limit is strictly positive at $b = 0$. For $b = \bar{b}_B$, we have

$$\begin{aligned} & \bar{A}(\bar{b}_B) - \frac{\delta(q^* - \underline{q})\bar{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\bar{b}_B \\ &= \frac{y - [r + \delta(1 - \underline{q})]\bar{b}_B + \lambda\bar{V}}{\rho + \lambda} - \frac{y - [r + \delta(1 - \underline{q})]\bar{b}_B}{\rho} + \frac{\delta(q^* - \underline{q})\bar{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\bar{b}_B \\ &= \frac{-\lambda}{\rho(\rho + \lambda)} \left(y - [r + \delta(1 - q^*)]\bar{b}_B - \rho\bar{V} \right) = \frac{-\lambda}{\rho(\rho + \lambda)} \left(r(\underline{b}_S - \bar{b}_B) - \delta(1 - q^*)\bar{b}_B \right) > 0, \end{aligned}$$

where the last inequality uses $\bar{b}_B > \underline{b}_S$. This completes the proof of part (i).

For part (ii), the saving equilibrium requires $V_B^P(\underline{b}_S) \leq \bar{V}$. As $\bar{C} \rightarrow \infty$,

$$\begin{aligned} V_B^P(\underline{b}_S) &= V_B^P(\bar{b}_B) + q^*(\bar{b}_B - \underline{b}_S) \\ &= \frac{y - r\bar{b}_B + \lambda\bar{V}}{\rho + \lambda} + \bar{b}_B - \underline{b}_S - (1 - q^*)(\bar{b}_B - \underline{b}_S) - \frac{\delta(1 - q^*)\bar{b}_B}{\rho + \lambda} \\ &= \bar{V} + \frac{(\rho + \lambda - r)(\bar{b}_B - \underline{b}_S)}{\rho + \lambda} - (1 - q^*) \left(\bar{b}_B - \underline{b}_S + \frac{\delta\bar{b}_B}{\rho + \lambda} \right). \end{aligned}$$

As the second term is strictly positive, there exists a $\tilde{q} < 1$ such that this expression exceeds \bar{V} for $q^* > \tilde{q}$, hence violating the necessary condition for a saving equilibrium. \square

B.12 Proof of Proposition 11

Proof. For part (i), note that in the saving equilibrium undistorted by policy, $q_S(b) = 1$ for $b \leq \underline{b}_S$. Hence, imposing a price floor restricted to the Safe Zone does not alter the saving equilibrium, which exists by Proposition 9. Hence, the price floor is irrelevant under the saving equilibrium.

Using the notation introduced in the proof of Proposition 10, a necessary condition for the borrowing equilibrium under the policy is for $b \in [0, \underline{b}_B^P]$

$$V_B^P(b) \geq \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \geq \frac{y - [r + \delta(1 - q^*)]b}{\rho}.$$

Recall that in the construction of the borrowing equilibrium, \underline{b}_B is defined by solving the HJB assuming

$q_B(b) = \underline{q}$. Hence, $V_B^P(b) = V_B(b)$ for $b > \underline{b}_S$, as the policy is restricted to $b \in [0, \underline{b}_S]$. As $V_B(\underline{b}_S) < \bar{V}$ by the inequality of Proposition 9, we have $\underline{b}_B^P < \underline{b}_S$. For $b = \underline{b}_B^P$, we have

$$V_B^P(\underline{b}_B^P) = \bar{V} = \frac{y - r\underline{b}_S}{\rho} < \frac{y - r\underline{b}_B^P}{\rho},$$

where the first two equalities use the definitions of \underline{b}_B^P and \underline{b}_S , respectively. Hence, there exists a $\hat{q} < 1$ such that

$$V_B^P(\underline{b}_B^P) < \frac{y - [r + \delta(1 - q^*)]\underline{b}_B^P}{\rho},$$

for $q^* > \hat{q}$, violating the necessary condition for a borrowing equilibrium. This proves part (ii). □

Online Appendix to “Self-Fulfilling Debt Dilution”

By Mark Aguiar and Manuel Amador

Appendix C: Viscosity Solutions on Stratified Domains and the Proofs of Propositions 1 and 4

In this appendix, we establish the equivalence between the sequence problems and the viscosity solutions of the Hamilton-Jacobi-Bellman (HJB) equations. The two complications are that the objective and/or the dynamics are not necessarily continuous in the state variables. We rely on the results of Bressan and Hong (2007) (henceforth, BH) to establish the validity of the recursive formulation. This appendix introduces their environment and summarizes their core results. Relative to BH, we make minor changes in notation and consider a maximization problem while the original BH studies minimization. We then prove Propositions 1 and 4.

C.1 The Environment of Bressan and Hong (2007)

Let $X \subset \mathbb{R}^N$ denote the state space. In the benchmark BH environment, $X = \mathbb{R}^N$; however, they show how to restrict attention to an arbitrary subset by extending the dynamics and payoff functions to \mathbb{R}^N such that the subset is an absorbing region. Let $\alpha(t) \in \mathcal{A}$ denote the control function, where \mathcal{A} is the set of admissible controls. Dynamics of the state vector x are given by $\dot{x}(t) = f(x(t), \alpha(t))$.

Given a discount factor β and a flow payoff $\ell(x, \alpha)$, the sequence problem is

$$W(\bar{x}) = \sup_{\alpha \in \mathcal{A}} \int_0^{\infty} \ell(x(t), \alpha(t)) dt \quad (57)$$

subject to $\begin{cases} x(0) &= \bar{x} \in X \\ \dot{x}(t) &= f(x(t), \alpha(t)). \end{cases}$

The complication BH address is that f may not be continuous in x . In particular, assume there is a decomposition $X = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_M$ with the following properties. If $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_k = \emptyset$. In addition, if $\mathcal{M}_j \cap \overline{\mathcal{M}_k} \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}_k}$, where $\overline{\mathcal{M}_k}$ is the closure of \mathcal{M}_k .

BH’s assumption **H1** ensures that dynamics are well behaved within \mathcal{M}_i :

Assumption. H1: For each $i = 1, \dots, M$, there exists a compact set of controls $A_i \subset \mathbb{R}^m$, a continuous map $f_i : \mathcal{M}_i \times A_i \rightarrow \mathbb{R}^N$, and a payoff function ℓ_i , with the following properties:

- (a) At each $x \in \mathcal{M}_i$, all vectors $f_i(x, a)$, $a \in A_i$ are tangent to \mathcal{M}_i ;
- (b) $|f_i(x, a) - f_i(z, a)| \leq K_i|x - z|$, for some $K_i \in [0, \infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$;
- (c) Each payoff function ℓ_i is non-positive and $|\ell_i(x, a) - \ell_i(z, a)| \leq L_i|x - z|$, for some $L_i \in [0, \infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$,³¹
- (d) We have $f(x, a) = f_i(x, a)$ and $\ell(x, a) = \ell_i(x, a)$ whenever $x \in \mathcal{M}_i$, $i = 1, \dots, M$.

³¹We strengthen part (c) to incorporate the Lipschitz continuity condition stated in BH equation (46).

The key assumption is (b); namely, that dynamics are Lipschitz continuous when confined to tangent trajectories. This does not restrict how the dynamics change when crossing the boundaries of \mathcal{M}_i .

Let $\mathcal{T}_{\mathcal{M}_i}(x)$ denote the cone of trajectories tangent to \mathcal{M}_i at $x \in \mathcal{M}_i$:

$$\mathcal{T}_{\mathcal{M}_i}(x) \equiv \left\{ y \in \mathbb{R}^N \mid \lim_{h \rightarrow 0} \frac{\inf_{z \in \mathcal{M}_i} |x + hy - z|}{h} = 0 \right\}.$$

Part (a) of **H1** is equivalent to $f_i(x, a) \in \mathcal{T}_{\mathcal{M}_i}$ for all $x \in \mathcal{M}_i, a \in A_i$.

For $x \in \mathcal{M}_i$, let $\hat{F}(x) \subset \mathbb{R}^{N+1}$ denote the set of achievable dynamics and payoffs for the set of controls A_i :

$$\hat{F}(x) \equiv \{(\dot{x}, u) \mid \dot{x} = f_i(x, a), u \leq \ell_i(x, a), a \in A_i\}, \quad (58)$$

where i is such that $x \in \mathcal{M}_i$. To handle discontinuous dynamics, BH use results from differential inclusions. In particular, let $G(x)$ denote an extended set of feasible trajectories and payoffs:

$$G(x) \equiv \cap_{\epsilon > 0} \overline{\text{co}} \{(\dot{x}, u) \in \hat{F}(x') \mid |x - x'| < \epsilon\}, \quad (59)$$

where $\overline{\text{co}}S$ denotes the closed convex hull of a set S .

The next key assumption is that $G(x)$ does not contain additional trajectory-payoff pairs when restricted to tangent trajectories:

Assumption. H2: For every $x \in \mathbb{R}^N$, we have

$$\hat{F}(x) = \{(\dot{x}, u) \in G(x) \mid \dot{x} \in \mathcal{T}_{\mathcal{M}_i}\}. \quad (60)$$

BH define the Hamiltonian using $G(x)$ as the relevant choice set:

$$H(x, p) \equiv \sup_{(\dot{x}, u) \in G(x)} \{u + p\dot{x}\}. \quad (61)$$

The corresponding HJB is

$$\beta w(x) = H(x, Dw(x)), \quad (62)$$

where D is the differential operator. BH define the following concepts:

Definition 3. A continuous function w is a **lower solution** of (62) if the following holds: If $w - \varphi$ has a local maximum at x for some continuously differential φ , then

$$\beta w(x) - H(x, D\varphi(x)) \leq 0. \quad (63)$$

Definition 4. A continuous function w is an **upper solution** of (62) if the following holds: If $x \in \mathcal{M}_i$, and the restriction of $w - \varphi$ to \mathcal{M}_i has a local minimum at x for some continuously differential φ , then

$$\beta w(x) - \sup_{(\dot{x}, u) \in G(x), \dot{x} \in T_{\mathcal{M}_i}} \{u + D\varphi\dot{x}\} \geq 0. \quad (64)$$

Definition 5. A continuous function w , which is both an upper and a lower solution of (62), is a **viscosity solution**.

Note that the second definition differs from the first by restricting attention to \mathcal{M}_i when describing the properties of $w - \varphi$, which relaxes the set of φ that satisfies the condition. However, the trajectories

in the Hamiltonian are restricted to lie in the tangent set.³² The added properties are the core distinction between the definition of viscosity solution used here versus the standard definition.³³

With these definitions in hand, we summarize the core results of BH:

- (i) (BH Theorem 1) If **H1** and **H2** hold, and there exists at least one trajectory with finite value, then the maximization problem admits an optimal solution.
- (ii) (BH Proposition 1) Let assumptions **H1** and **H2** hold. If the value function W is continuous, then it is a viscosity solution of (62).
- (iii) (BH Corollary 1) Let assumptions **H1** and **H2** hold. If the value function W is bounded and Lipschitz continuous, then W is the unique non-positive viscosity solution to (62).³⁴

C.2 The Planner's Problem

To map problem (3) into BH, we make a few modifications and consider a generalized problem. First, we let the planner randomize when the government is indifferent to default or not. This helps to convexify the choice set. In particular, let $\pi(t) \in [0, 1]$ be an additional choice, where $\pi(t)$ is the probability the government defaults when \bar{V} arises and the current value is \bar{V} . It will always be efficient to set $\pi(t) = 0$ when $v(t) = \bar{V}$, and so this does not alter the solution to the planner's problem. We denote the set of possible paths, $\pi = \{\pi(t) \in [0, 1]\}_{t \geq 0}$, by Π . The controls are $\alpha = (c, \pi) \in \mathcal{A} \equiv \mathcal{C} \times \Pi$.

Recall that in (3) the objective is discounted by the probability of repayment, $e^{-\lambda \int_0^t \mathbb{1}_{[v(s) < \bar{V}]} ds}$. Let us define $\Gamma(t)$ as follows:

$$\Gamma(t) \equiv \Gamma(0) e^{-\lambda \int_0^t (\pi(s) \mathbb{1}_{[v(s) = \bar{V}]} + \mathbb{1}_{[v(s) < \bar{V}]} ds)}$$

for some $\Gamma(0) \in (0, 1]$. Note that $\Gamma(t)/\Gamma(0)$ is the discount factor in the original problem with $\pi = 0$. By adding $\Gamma(t)$ as an additional state variable, we will be able to keep track of the probability of survival in our recursive formulation.

Let $X = \mathbb{V} \times (0, 1]$ denote the state space for $x = (v, \Gamma)$. Let $f(x, \alpha) = (\dot{v}, \dot{\Gamma})$ given the control $\alpha = (c, \pi)$:

$$f(x, \alpha) = \begin{cases} \dot{v} & = -c + \rho v - \mathbb{1}_{[v < \bar{V}]} \lambda [\bar{V} - v] \\ \dot{\Gamma} & = -\lambda \left[\pi \mathbb{1}_{[v = \bar{V}]} + \mathbb{1}_{[v < \bar{V}]} \right] \Gamma. \end{cases} \quad (65)$$

The flow value must be non-positive. We therefore subtract the constant $(y - \underline{C})/r$ from the value. To convert this into a flow payoff, let

$$\ell(x, a) \equiv \Gamma(y - c) - (y - \underline{C}) \leq 0,$$

where the inequality uses $y > \underline{C}$ and $\Gamma \leq 1$. Note that ℓ is Lipschitz continuous in x .

³²The fact that trajectories are restricted to T_{M_i} in the definition of an upper solution was unintentionally omitted in Bressan and Hong (2007) but is corrected in Bressan (2013).

³³Note that we place the restriction on the upper solution while the original BH place it on the lower solution as we consider a maximization problem.

³⁴BH state a weaker continuity condition than Lipschitz continuity (BH **H3**) that is not necessary given our environment.

Hence, we consider the following problem, where $x(t) \equiv (v(t), \Gamma(t))$:

$$\begin{aligned} \tilde{P}(v, \Gamma) &= \sup_{\alpha \in \mathcal{A}} \int_0^\infty e^{-rt} \ell(x(t), \alpha(t)) dt \\ &\text{subject to } \begin{cases} (v(0), \Gamma(0)) &= (v, \Gamma) \\ (\dot{v}(t), \dot{\Gamma}(t)) &= f(x(t), \alpha(t)). \end{cases} \end{aligned} \quad (66)$$

We then have a one-to-one mapping between \tilde{P} and P^\star :

$$\tilde{P}(v, \Gamma) = \Gamma P^\star(v) - (y - \underline{C})/r. \quad (67)$$

As P^\star is bounded and $\Gamma \in (0, 1]$, \tilde{P} is bounded. Similarly, \tilde{P} is Lipschitz continuous in the state vector (v, Γ) .

We now verify the conditions of BH. Define five regions of the state space:

$$\begin{aligned} \mathcal{M}_1 &\equiv \{\underline{V}\} \times (0, 1] \\ \mathcal{M}_2 &\equiv (\underline{V}, \bar{V}) \times (0, 1] \\ \mathcal{M}_3 &\equiv \{\bar{V}\} \times (0, 1] \\ \mathcal{M}_4 &\equiv (\bar{V}, V_{max}) \times (0, 1] \\ \mathcal{M}_5 &\equiv \{V_{max}\} \times (0, 1]. \end{aligned}$$

Let A_i denote the controls that produce trajectories that are tangent to \mathcal{M}_i :³⁵

$$\begin{aligned} A_i &\equiv \{(c, \pi) | c \in [\underline{C}, \bar{C}], \pi \in [0, 1], \dot{x} \in \mathcal{T}_{\mathcal{M}_i}\} \\ &= \begin{cases} \{\rho \underline{V} - \lambda(\bar{V} - \underline{V})\} \times [0, 1] &\text{if } i = 1 \\ \{\rho \bar{V}\} \times [0, 1] &\text{if } i = 3 \\ \{\rho V_{max}\} \times [0, 1] &\text{if } i = 5 \\ [\underline{C}, \bar{C}] \times [0, 1] &\text{otherwise.} \end{cases} \end{aligned} \quad (68)$$

Within each \mathcal{M}_i , the dynamics f are Lipschitz continuous in x for all $a \in A_i$. It is straightforward to verify that we satisfy Assumption **H1**.

Let us now verify Assumption **H2**. There two cases:

Case 1: $i \in \{2, 4\}$. In this case, $G(x) = \hat{F}(x)$, and BH Assumption **H2** is straightforward to verify.

Case 2: $i \in \{1, 3, 5\}$. We show the $i = 3$ case (as the others are similar). We have³⁶

$$\hat{F}(x) = \left\{ (\dot{x}, u) | \dot{v} = 0, \dot{\Gamma} = -\pi \lambda \Gamma, u \leq \ell(x, (\rho \bar{V}, \pi)), \pi \in [0, 1] \right\} \quad (69)$$

$$\begin{aligned} &= \left\{ (\dot{x}, u) | \dot{v} = 0, \dot{\Gamma} \in [-\lambda \Gamma, 0], u \leq \Gamma(y - \rho \bar{V}) - (y - \underline{C}) \right\} \\ &= \{0\} \times [-\lambda \Gamma, 0] \times (-\infty, \Gamma(y - \rho \bar{V}) - (y - \underline{C})). \end{aligned} \quad (70)$$

³⁵For $i = 1, 3, 5$, the tangent trajectories set $\dot{v} = 0$. Otherwise, they are the full set of trajectories.

³⁶Note this is the only case where the choice of π is relevant.

Let $x' = (v', \Gamma')$ be in the neighborhood of $x = (\bar{V}, \Gamma)$. We have

$$\begin{aligned}\hat{F}(x') &= \{(\dot{x}, u) \mid \\ &\quad \dot{v} = -c + \rho v' - \lambda \mathbb{1}_{\{v' < \bar{V}\}}(\bar{V} - v'), \\ &\quad \dot{\Gamma} \in [-\lambda \mathbb{1}_{\{v' < \bar{V}\}} \Gamma, 0], \\ &\quad u \leq \Gamma(y - c) - (y - \underline{C}), c \in [\underline{C}, \bar{C}]\}.\end{aligned}$$

We have that

$$\begin{aligned}\cup_{|x'-x| \leq \epsilon} \hat{F}(x') \subseteq R(x, \epsilon) \equiv & \left\{ \begin{aligned} \dot{v} &= -c + \theta, \\ \dot{\Gamma} &= [-\lambda(\Gamma + \epsilon), 0], \\ u &\leq (\Gamma + \epsilon - 1)y - (\Gamma - \epsilon)c + \underline{c}, \\ \theta &\in [\rho(\bar{V} - \epsilon) - \lambda\epsilon, \rho(\bar{V} + \epsilon)] \\ c &\in [\underline{C}, \bar{C}]. \end{aligned} \right.\end{aligned}$$

Note that $R(x, \epsilon)$ is convex and $G(x) = \cap_{\epsilon > 0} R(x, \epsilon)$. Also note that

$$\hat{F}(x) = \{(\dot{x}, u) \in G(x) \mid \dot{x} \in \mathcal{T}_{\mathcal{M}_3}\},$$

where the equivalence uses the definitions of G , \hat{F} , and the tangent trajectories $\mathcal{T}_{\mathcal{M}_3}$. This verifies BH **H2** for \mathcal{M}_3 .

Similar steps hold for $i = 1$ and 5 , verifying Assumption **H2** for all domains.³⁷

As noted above, \tilde{P} is bounded and Lipschitz continuous. Hence, by BH Corollary 1, it is the unique viscosity solution with such properties for the HJB:

$$r\tilde{P}(v, \Gamma) = H((v, \Gamma), (\tilde{P}_v, \tilde{P}_\Gamma)) \equiv \sup_{(c, \pi) \in [\underline{C}, \bar{C}] \times [0, 1]} \{ \Gamma(y - c) - (y - \underline{C}) + \tilde{P}_v \dot{v} + \tilde{P}_\Gamma \dot{\Gamma} \}, \quad (71)$$

where \dot{v} and $\dot{\Gamma}$ obey equation (65). Here, we have used the fact that $G(x)$ contains the full set of trajectories generated by $c \in [\underline{C}, \bar{C}]$ and $\pi \in [0, 1]$. Note that it is optimal to set π to 0, and thus we can ignore this choice in the Hamiltonian in what follows. We shall use the fact that H is convex in \tilde{P}_v .

C.3 Proof of Proposition 1

Proof. Suppose that $p(v)$ satisfies the conditions in the proposition. We shall show that $\tilde{p}(v, \Gamma) \equiv \Gamma p(v) - (y - \underline{C})/r$ is a viscosity solution of (71). \tilde{p} is differentiable in Γ at all points, and in v at points where $p(v)$ is differentiable. We now check the conditions for a viscosity solution. We proceed by checking on each domain \mathcal{M}_i .

- (i) $(v, \Gamma) \in \mathcal{M}_2 \cup \mathcal{M}_4$

³⁷For $v = \underline{V}$, we extend the dynamics to both sides of \underline{V} by setting $\dot{v} = -c + \rho v - \lambda(\bar{V} - v)$ in the neighborhood $v < \underline{V}$ and ℓ arbitrarily low. Thus, the dynamics are continuous at $x = (\underline{V}, \Gamma)$. Similarly for $v = V_{max}$, we set $\dot{v} = -c + \rho v$.

As p is differentiable on this part of the domain, by condition (i) of the proposition, we have

$$\begin{aligned} rp(v) &= \sup_{c \in [\underline{C}, \bar{C}]} \left\{ y - c + p'(v)\dot{v} + \mathbb{1}_{[v < \bar{V}]} p(v) \right\} \\ &= \sup_{c \in [\underline{C}, \bar{C}]} \left\{ y - c + \Gamma^{-1} \tilde{p}_v \dot{v} + \Gamma^{-1} \tilde{p}_\Gamma \dot{\Gamma} \right\}, \end{aligned}$$

where the second line uses $\tilde{p}_v = \Gamma p'(v)$ and $\tilde{p}_\Gamma \dot{\Gamma} / \Gamma = -\lambda \mathbb{1}_{[v < \bar{V}]} p$. Multiplying through by $\Gamma \in (0, 1]$ and subtracting $(y - \underline{C})/r$ from both sides yields

$$\begin{aligned} r\tilde{p}(v) &= r(\Gamma p(v) - (y - \underline{C})) = \sup_{c \in [\underline{C}, \bar{C}]} \left\{ \Gamma(y - c) - (y - \underline{C})/r + \tilde{p}_v \dot{v} + \tilde{p}_\Gamma \dot{\Gamma} \right\} \\ &= H((v, \Gamma), (\tilde{p}_v, \tilde{p}_\Gamma)). \end{aligned}$$

Hence, \tilde{p} satisfies (71).

Now consider a point of non-differentiability \tilde{v} . As $(v, \Gamma) \notin \mathcal{M}_3$, $\tilde{v} \neq \bar{V}$, and hence condition (iii) of the proposition is applicable. Condition (iii) states that $p_{\tilde{v}}^- \equiv \lim_{v \uparrow \tilde{v}} p'(v) < \lim_{v \downarrow \tilde{v}} p'(v) \equiv p_{\tilde{v}}^+$. Hence, there is a convex kink. In this case, the lower solution does not impose additional conditions, leaving the conditions for an upper solution to be verified. Suppose φ is differentiable and $\tilde{p} - \varphi$ has a local minimum at (\tilde{v}, Γ) . Then $\varphi_v \in [p_{\tilde{v}}^-, p_{\tilde{v}}^+]$. As \tilde{p} is differentiable in Γ , we have $\varphi_\Gamma = \tilde{p}_\Gamma$. Note that

$$r\tilde{p}(\tilde{v}) = H((\tilde{v}, \Gamma), (p_{\tilde{v}}^-, \tilde{p}_\Gamma)) = H((\tilde{v}, \Gamma), (p_{\tilde{v}}^+, \tilde{p}_\Gamma)), \quad (72)$$

as the HJB holds with equality at points of differentiability in the neighborhood of \tilde{v} , and using the continuity of H .

Note that there exists $\alpha \in [0, 1]$ such that $\varphi_v = \alpha p_{\tilde{v}}^+ + (1 - \alpha)p_{\tilde{v}}^-$. The convexity of H with respect to φ_v implies that

$$\begin{aligned} H((\tilde{v}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= H((\tilde{v}, \Gamma), (\alpha p_{\tilde{v}}^+ + (1 - \alpha)p_{\tilde{v}}^-, \varphi_\Gamma)) \\ &\leq \alpha H((\tilde{v}, \Gamma), (p_{\tilde{v}}^+, \varphi_\Gamma)) + (1 - \alpha)H((\tilde{v}, \Gamma), (p_{\tilde{v}}^-, \varphi_\Gamma)) \\ &= r\tilde{p}(\tilde{v}), \end{aligned}$$

where the last equality uses (72) and $\varphi_\Gamma = \tilde{p}_\Gamma$. Hence, $\tilde{p}(\tilde{v})$ satisfies the conditions of an upper solution.

(ii) $(v, \Gamma) \in \mathcal{M}_3 = \{\bar{V}\} \times (0, 1]$

Turning to $v = \bar{V}$, we redefine $p_v^+ \equiv \lim_{v \downarrow \bar{V}} p'(v)$ and $p_v^- \equiv \lim_{v \uparrow \bar{V}} p'(v)$. Given the continuity of p and the fact that it satisfies the HJB in the neighborhood of \bar{V} with equality, we have

$$\begin{aligned} rp(\bar{V}) &= \sup_{c \in [\underline{C}, \bar{C}]} \{y - c + p_v^+ \dot{v}\} \\ &= \sup_{c \in [\underline{C}, \bar{C}]} \{y - c + p_v^- \dot{v} - \lambda p(\bar{V})\}, \end{aligned} \quad (73)$$

where $\dot{v} = -c + \rho \bar{V}$. As setting $c = \rho \bar{V}$ is always feasible, this implies $rp(\bar{V}) \geq (y - \rho \bar{V}) \geq 0$.

To verify that \tilde{p} is a viscosity solution to (61), note that if \tilde{p} is differentiable, then it satisfies the HJB with equality by condition (i) of the proposition.

If it is not differentiable, we consider convex and concave kinks in turn.

Suppose that $p_v^- < p_v^+$. Then the conditions for a lower solution do not impose any restrictions. For an upper solution, consider a φ such that $\tilde{p} - \varphi$ has a local minimum at (\bar{V}, Γ) . Again, $\varphi_\Gamma = \tilde{p}_\Gamma = p(\bar{V})$. Recall that for an upper solution, we need only consider trajectories that are in $\mathcal{T}_{\mathcal{M}_3}$, that is, $\dot{v} = 0$ and thus $c = \rho\bar{V}$. Hence:

$$\begin{aligned} r\tilde{p}(\bar{V}) &= r\Gamma p(\bar{V}) - (y - \underline{C}) \\ &\geq \Gamma(y - \rho\bar{V}) - (y - \underline{C}) \\ &= \sup_{c=\rho\bar{V}} \left\{ \Gamma(y - c) - (y - \underline{C}) + \varphi_v \underbrace{(-c + \rho\bar{V})}_{\dot{v}} \right\} \\ &= \sup_{c=\rho\bar{V}, \pi \in [0, 1]} \left\{ \Gamma(y - c) - (y - \underline{C}) + \varphi_v \underbrace{(-c + \rho\bar{V})}_{\dot{v}} - \underbrace{p(\bar{V}) \times \pi \lambda \Gamma}_{\varphi_\Gamma \times \dot{\Gamma}} \right\}, \end{aligned}$$

where the last equality uses $p(\bar{V}) \geq 0$. Note that final term is the Hamiltonian maximized along tangent trajectories in $\mathcal{T}_{\mathcal{M}_3}$. Thus, the conditions of an upper solution are satisfied.

For the lower solution, we must consider the case in which $\tilde{p} - \varphi$ has a local maximum at (\bar{V}, Γ) . This requires $p_v^- \geq p_v^+$ and $\varphi_v \in [p_v^+, p_v^-]$. Again, as \tilde{p} is differentiable with respect to Γ , we have $\varphi_\Gamma = \tilde{p}_\Gamma = p(\bar{V})$.

If $p_v^+ \leq -1$, then condition (ii) in the proposition implies that

$$\begin{aligned} r\tilde{p}(\bar{V}) &= \Gamma(y - \rho\bar{V}) - (y - \bar{C}) \\ &\leq \sup_{c \in [\underline{C}, \bar{C}], \pi \in [0, 1]} \left\{ \Gamma(y - c) - (y - \bar{C}) + \varphi_v \underbrace{(-c + \rho\bar{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi \lambda \Gamma)}_{\dot{\Gamma}} \right\} \\ &= H((\bar{V}, \Gamma), (\varphi_v, \varphi_\Gamma)), \end{aligned}$$

where the second to the last line follows from $\varphi_\Gamma = p(\bar{V}) \geq 0$. Hence, $\tilde{p}(\bar{V}) = \Gamma p(\bar{V}) - (y - \underline{C})/r$ satisfies the condition for a lower solution when $p_v^+ \leq -1$.

Alternatively, if $p_v^+ > -1$, then

$$\begin{aligned} rp(\bar{V}) &= \sup_{c \in [\underline{C}, \bar{C}]} \{y - c + p_v^+(-c + \rho\bar{V})\} \\ &= y - \underline{C} + p_v^+(\rho\bar{V} - \underline{C}) \\ &\leq y - \underline{C} + \varphi_v(\rho\bar{V} - \underline{C}), \end{aligned}$$

for $\varphi_v \geq p_v^+$ as $\rho\bar{V} > \underline{C}$. Hence,

$$r\tilde{p}(\bar{V}) \leq \sup_{c \in [\underline{C}, \bar{C}], \pi \in [0, 1]} \left\{ \Gamma(y - c) - (y - \bar{C}) + \varphi_v \underbrace{(-c + \rho\bar{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi \lambda \Gamma)}_{\dot{\Gamma}} \right\}$$

for $\varphi_v \in [p_v^+, p_v^-]$ and $\varphi_\Gamma = p(\bar{V})$, satisfying the condition for a lower solution.

(iii) $(v, \Gamma) \in \mathcal{M}_1 = \{\underline{V}\} \times (0, 1]$

For $v = \underline{V}$, the condition for \tilde{p} to be an upper solution is

$$r\tilde{p}(\underline{V}, \Gamma) \geq \Gamma \left(y - \rho\underline{V} + \lambda(\bar{V} - \underline{V}) \right) - (y - \underline{C}) - \lambda p(\underline{V})\Gamma,$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v} = 0$. As \tilde{p} satisfies the HJB with equality in the neighborhood of \underline{V} , we have

$$\begin{aligned} p(\underline{V}, \Gamma) &= \lim_{v \downarrow \underline{V}} r\tilde{p}(v, \Gamma) = \lim_{v \downarrow \underline{V}} H((v, \Gamma), (\Gamma p'(v), p(v))) \\ &\geq \lim_{v \downarrow \underline{V}} \left\{ \Gamma \left(y - \rho v + \lambda(\bar{V} - v) \right) - (y - \underline{C}) - \lambda p(v)\Gamma, \right\} \\ &= \Gamma \left(y - \rho\underline{V} + \lambda(\bar{V} - \underline{V}) \right) - (y - \underline{C}) - \lambda p(\underline{V})\Gamma. \end{aligned}$$

Hence, \tilde{p} is an upper solution.

Turning to the lower solution, suppose $\tilde{p} - \varphi$ has a local maximum at (\underline{V}, Γ) . As \underline{V} is at the boundary of the state space, this implies $\varphi_v \geq \tilde{p}_v(\underline{V}, \Gamma)$ and $\varphi_\Gamma = p(\underline{V})$. A lower solution requires

$$r\tilde{p}(\underline{V}, \Gamma) \leq H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)).$$

Suppose $p'(\underline{V}) < -1$. Then, condition (iv) of the proposition implies

$$\begin{aligned} r\tilde{p}(\underline{V}, \Gamma) &= r\Gamma p(\underline{V}) - (y - \underline{C}) \\ &= r\Gamma \left(\frac{y - \rho\underline{V} + \lambda(\bar{V} - \underline{V})}{r + \lambda} \right) - (y - \underline{C}) \\ &= \left(1 - \frac{\lambda}{r + \lambda} \right) \Gamma \left(y - \rho\underline{V} + \lambda(\bar{V} - \underline{V}) \right) - (y - \underline{C}) \\ &\leq \sup_{c \in [\underline{C}, \bar{C}]} \left\{ \Gamma(y - c) - (y - \bar{C}) + \varphi_v \underbrace{(-c + \rho\underline{V} - \lambda(\bar{V} - \underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda\Gamma)}_{\dot{\Gamma}} \right\} \\ &= H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)), \end{aligned}$$

where the inequality uses

$$\begin{aligned} -\varphi_\Gamma \lambda \Gamma &= -p(\underline{V})\lambda\Gamma \\ &= - \left(y - \rho\underline{V} + \lambda(\bar{V} - \underline{V}) \right) \frac{\lambda}{r + \lambda} \Gamma. \end{aligned}$$

This verifies that \tilde{p} is a lower solution if $p'(\underline{V}) < -1$.

Turning to $p'(\underline{V}) \geq -1$,

$$\begin{aligned}
H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= \\
&= \sup_{c \in [\underline{C}, \bar{C}]} \left\{ \Gamma(y - c) - (y - \bar{C}) + \varphi_v \underbrace{(-c + \rho \underline{V} - \lambda(\bar{V} - \underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda \Gamma)}_{\dot{\Gamma}} \right\} \\
&= \Gamma(y - \underline{C}) - (y - \bar{C}) + \varphi_v \underbrace{(-\underline{C} + \rho \underline{V} - \lambda(\bar{V} - \underline{V}))}_{>0} + \varphi_\Gamma(-\lambda \Gamma) \\
&\geq \Gamma(y - \underline{C}) - (y - \bar{C}) + \tilde{p}_v(\underline{V}, \Gamma)(-\underline{C} + \rho \underline{V} - \lambda(\bar{V} - \underline{V})) + \varphi_\Gamma(-\lambda \Gamma) \\
&= H((\underline{V}, \Gamma), (\tilde{p}_v, \tilde{p}_\Gamma)) = r\tilde{p}(\underline{V}, \Gamma),
\end{aligned}$$

where the second equality uses the fact that \underline{C} is optimal when $\varphi_v \geq \Gamma p'(v) \geq -\Gamma$; the inequality uses the fact that $\varphi_v \geq \tilde{p}_v$ and the term multiplying φ_v is positive; and the last line uses the continuity of the Hamiltonian and the value function, and that \underline{C} is optimal given $p'(\underline{V}) \geq -1$. This verifies that \tilde{p} is a lower solution if $p'(\underline{V}) \geq -1$.

(iv) $(v, \Gamma) \in \mathcal{M}_5 = \{V_{max}\} \times (0, 1]$

For $v = V_{max}$, the condition for \tilde{p} to be an upper solution is

$$r\tilde{p}(V_{max}, \Gamma) \geq \Gamma(y - \rho V_{max}) - (y - \underline{C}),$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v} = 0$. As \tilde{p} satisfies the HJB with equality in the neighborhood of V_{max} , we have

$$\begin{aligned}
r\tilde{p}(V_{max}, \Gamma) &= \lim_{v \uparrow V_{max}} r\tilde{p}(v, \Gamma) = \lim_{v \uparrow V_{max}} H((v, \Gamma), (\Gamma p'(v), p(v))) \\
&\geq \lim_{v \uparrow V_{max}} \{ \Gamma(y - \rho v) - (y - \underline{C}) \} \\
&= \Gamma(y - \rho V_{max}) - (y - \underline{C}).
\end{aligned}$$

Hence, \tilde{p} is an upper solution.

For the lower solution, suppose $\tilde{p} - \varphi$ has a local maximum at (V_{max}, Γ) . This implies $\varphi_v \leq \tilde{p}_v = \Gamma p'(V_{max})$ and $\varphi_\Gamma = p(V_{max})$. The condition for a lower solution is

$$\begin{aligned}
r\tilde{p}(V_{max}, \Gamma) &\leq \sup_{c \in [\underline{C}, \bar{C}]} \left\{ \Gamma(y - c) - (y - \bar{C}) + \varphi_v \underbrace{(-c + \rho V_{max})}_{\dot{v}} \right\} \\
&= H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)).
\end{aligned}$$

By condition (v) of the proposition, we have $p'(V_{max}) \leq -1$, implying that $\varphi_v \leq -\Gamma$. Hence, $c = \bar{C}$

achieves the optimum in $H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma))$. That is,

$$\begin{aligned} H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= \Gamma(y - \bar{C}) - (y - \bar{C}) + \varphi_v \underbrace{(-\bar{C} + \rho V_{max})}_{\dot{v}} \\ &\geq \Gamma(y - \bar{C}) - (y - \bar{C}) + \tilde{p}_v(-\bar{C} + \rho V_{max}) \\ &= r\tilde{p}(V_{max}, \Gamma), \end{aligned}$$

where the inequality uses $\varphi_v \geq \tilde{p}_v$ and $\bar{C} \geq \rho V_{max}$; and the final line uses continuity of H and \tilde{p} . Hence, \tilde{p} is a lower solution at (V_{max}, Γ) .

We have shown that \tilde{p} implied by a p satisfying the conditions of the proposition is a viscosity solution of the planner's problem. \square

C.4 The Competitive Equilibrium

This section maps the government's problem into the BH framework.

Let us first define the following operator T that takes as an input a candidate value function, $\tilde{V}(b)$, assumed to be bounded and Lipschitz continuous, and a debt dynamics function $f(b, c)$ that embeds the price function, $q(b)$:

$$T\tilde{V}(b) = \int_0^\infty e^{-(r+\lambda)t} \left(c(t) + \lambda D(b(t)|\tilde{V}) \right) \quad (74)$$

subject to:

$$\begin{aligned} \dot{b}(t) &= f(b(t), c(t)) \\ b(0) &= b, \end{aligned}$$

where

$$D(b|\tilde{V}) \equiv \mathbb{1}_{[\tilde{V}(b) < \bar{V}]}(\bar{V} - \tilde{V}(b)).$$

The government's equilibrium value function is a fixed point of this operator. We shall map the right-hand side problem into the BH framework and use recursive techniques to solve the optimization. Toward this goal, let

$$\ell(b, c) \equiv c + \lambda D(b|\tilde{V}).$$

Note that $\ell(b, c)$ so defined is Lipschitz continuous and bounded. To be consistent with BH, we also need a non-positive ℓ . This can be achieved by subtracting the maximum value of ℓ . Rather than carrying this notation through, we proceed with the objects defined above, recognizing that all flow utilities and values can be appropriately translated (as we did explicitly in the planning problem).

Turning to the dynamics, $f(b, c)$, suppose the government faces a closed, convex domain \mathbf{B} and an equilibrium price schedule $q : \mathbf{B} \rightarrow [q, 1]$ that is differentiable almost everywhere with $|q'(b)| < M < \infty$.

Let $b_0 \equiv -\bar{a}$; b_1, \dots, b_N denote the N points of non-continuity in q ; and $b_{N+1} \equiv \bar{b}$. We consider equilibria in which $\limsup_{b \rightarrow b_n} q(b) = q(b_n)$, as our tie-breaking rule is that the government chooses the action that maximizes the price when indifferent.

To define the domains, let $\mathcal{M}_n \equiv (b_{n-1}, b_n)$, $n = 1, \dots, N+1$, be the open sets on which q is differentiable. Let $\mathcal{M}_{N+1+n} \equiv \{b_n\}$, $n = 1, \dots, N$ be the isolated points of non-differentiability. Finally, we have the endpoints of the domain: $\{-\bar{a}\}$ and $\{\bar{b}\}$.

In the neighborhood of a discontinuity, we rule out repurchases at the “low price” (see footnotes 19 and 23). We do this while ensuring the continuity of dynamics. Specifically, let $\Delta > 0$ be arbitrarily small; and in particular, $\Delta < \inf_n |b_n - b_{n-1}|/2$. Define $\alpha(b) \equiv \min\{|b - b_n|/\Delta, 1\}$, where b_n is the closest point of non-differentiability to b . Note that $\alpha(b) \in [0, 1]$, and equals one if $|b - b_n| \geq \Delta$. Debt dynamics are given by

$$f(b, c) = \begin{cases} \frac{c - y + (r + \delta)b}{q(b)} - \delta b & \text{if } c \geq y - (r + \delta)b \\ \frac{c - y + (r + \delta)b}{\alpha(b)q(b) + (1 - \alpha(b))q(b_n)} - \delta b & \text{if } c < y - (r + \delta)b. \end{cases} \quad (75)$$

Note that $f(b, c)$ is Lipschitz continuous in b and c within the domains \mathcal{M}_n .

On the open sets \mathcal{M}_n , $n = 1, \dots, N+1$, any $c \in A_n \equiv [\underline{C}, \bar{C}]$ results in a tangent trajectory. For $n > N+1$, $c \in A_n \equiv y - [r + \delta(1 - q(b_n))]b_n$ is the singleton set that generates a tangent trajectory to the isolated point \mathcal{M}_n . Hence, BH assumption **H1** is satisfied.

Following BH, define

$$\hat{F}(b) \equiv \{(\dot{b}, u) | \dot{b} = f(b, c), u \leq \ell(b, c), c \in A_n\}. \quad (76)$$

If $b = b_n$ for some n , we have

$$\hat{F}(b_n) = \{0\} \times \{u \leq \ell(b, y - [r + \delta(1 - q(b_n))]b_n)\}. \quad (77)$$

Otherwise,

$$\hat{F}(b) = \{(\dot{b}, u) | \dot{b} \in [f(b, \underline{C}), f(b, \bar{C})], u \leq \ell(b, q(b)(\dot{b} + \delta b) + y - (r + \delta)b)\}. \quad (78)$$

Finally, define

$$G(b) \equiv \cap_{\varepsilon > 0} \overline{\text{co}} \{(\dot{b}, u) \in \hat{F}(b') \text{ such that } |b' - b| < \varepsilon\}. \quad (79)$$

To characterize this set, if $b \neq b_n$, then $G(b) = \hat{F}(b)$ as f is continuous within the open set \mathcal{M}_n , $n = 1, \dots, N+1$, and the tangent trajectories are generated by $c \in [\underline{C}, \bar{C}]$. For $b = b_n$ for some n , we have

$$G(b_n) = \{(\dot{b}, u) | \dot{b} = f(b, c), u \leq \ell(b, c), c \in [\underline{C}, \bar{C}]\}.$$

For this case, restricting attention to $c = y - [r + \delta(1 - q(b_n))]b_n$ yields $\hat{F}(b_n)$. Hence BH assumption **H2** is satisfied.

We use the assumption regarding repurchases around a point of discontinuity in q to rule out the following. Suppose that the following trajectory was feasible: $\dot{b} < -\delta b$ and $c = \liminf_{b \rightarrow b_n} q(b_n)(\dot{b} - \delta b) - (r + \delta)b + y > q(b_n)(\dot{b} - \delta b) - (r + \delta)b + y$. Then, in the convexification generating $G(b_n)$, a trajectory featuring $\dot{b} = 0$ and $c > y - [r + \delta(1 - q(b_n))]b$ would appear. This new trajectory would be generated by locating two trajectories featuring $\dot{b} < -\delta b$ and $\dot{b} > -\delta b$, such that their convex combination leads to $\dot{b} = 0$. Because for the trajectory with $\dot{b} > -\delta b$ we have that $c = \bar{C}$, the associated convex combination of the consumptions of these two trajectories would then be strictly greater than the stationary consumption in $\hat{F}(b_n)$, violating **H2**.

BH Proposition 1 and Corollary 1 then imply that the solution to $T\tilde{V}$ is the unique bounded, Lipschitz

continuous viscosity solution to

$$\rho(T\tilde{V})(b) = \sup_{c \in [\underline{C}, \bar{C}]} \{c + \lambda D(b|\tilde{V}) + (T\tilde{V})'(b)f(b, c)\}.$$

As V is a fixed point of the operator, the government's value V is a viscosity solution to

$$\rho V(b) = H(b, V'(b)) \equiv \sup_{c \in [\underline{C}, \bar{C}]} \left\{ c + \lambda \mathbb{1}_{[V(b) < \bar{V}]} (\bar{V} - V(b)) + V'(b)f(b, c) \right\}, \quad (80)$$

where the term $\lambda \mathbb{1}_{[V(b) < \bar{V}]} (\bar{V} - V(b))$ is taken as a given function of b in verifying the viscosity conditions.

C.5 Proof of Proposition 4

Proof. We need to verify that if v satisfies the conditions of the proposition, it also satisfies the conditions for a viscosity solution. The proof and details parallel that of the proof for Proposition 1, and we omit some of the identical steps.

Lower solution conditions. In regard to the conditions for a lower solution, condition (i) in the proposition ensures these are met wherever v is differentiable on the interior of \mathbf{B} . At the boundaries, $-\bar{a}$ and \bar{b} , conditions (iv) and (v) of the proposition state that v equals the stationary value. Hence, $\rho v(b) \leq H(b, \varphi'(b))$, $b \in \{-\bar{a}, \bar{b}\}$, for any $\varphi'(b)$, as $\dot{b} = 0$ is always feasible.

For a non-differentiability at \underline{b} , the same argument as for $P(\bar{V})$ in the proof of Proposition 1 applies. That is, if v has a concave kink, then condition (ii) imposes that value must be the stationary value, which is (weakly) less than $H(b, \varphi'(b))$ for any $\varphi'(b)$. For a convex kink, the lower solution does not impose any restrictions.

At all other points of non-differentiability, condition (iii) states that v has a convex kink, and therefore $v - \varphi$ cannot have a local maximum for a smooth function φ . Thus, the lower solution does not impose any restrictions.

Upper solution conditions. For the upper solution, condition (i) of the proposition states that v satisfies the definition of an upper solution wherever it is differentiable. For points of non-differentiability at $\tilde{b} \neq \underline{b}$, first suppose that q is continuous at \tilde{b} . Condition (iii) guarantees that v has a convex kink at \tilde{b} , and as in the proof of Proposition 1, then the convexity of $H(b, \varphi'(b))$ in $\varphi'(b)$ ensures the upper solution inequality is satisfied. If q is not continuous at \tilde{b} , then the ‘‘tangent trajectories’’ are restricted to $\dot{b} = 0$. Hence, we need to check that v is weakly greater than the stationary value. This is satisfied by a continuity argument that parallels that in the proof of Proposition 1.

□